Phys 521 Fall 2023

# HW #11 Solutions

(1) a) $\Psi(x,t) = \exp\left[i(px - Et)/\hbar\right]$

$$\psi^*(x,-t) = \exp\left[-i(px + Et)/\hbar\right]$$

Constant phase $\Delta\phi = 0 = p\,\Delta x + E\,\Delta t$

velocity $\frac{\Delta x}{\Delta t} = -\frac{p}{E}$ particle moves in $-x$ direction

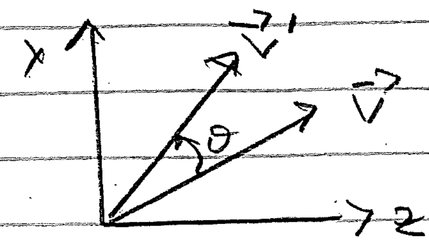
b) $\chi(\hat{n}) = d^{1/2}(\phi,\theta)\begin{pmatrix}1\\0\end{pmatrix}$

$$d^{1/2}(\phi,\theta) = \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 \,, & -e^{-i\phi/2}\sin\theta/2 \\ e^{i\phi/2}\sin\theta/2 \,, & e^{i\phi/2}\cos\theta/2 \end{pmatrix}$$

$$\chi(\hat{n}) = \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 \\ e^{i\phi/2}\sin\theta/2 \end{pmatrix} ; \quad \chi(-\hat{n}) = \begin{pmatrix} -e^{-i\phi/2}\sin\theta/2 \\ e^{i\phi/2}\cos\theta/2 \end{pmatrix}$$

$$-i\sigma_2\chi^*(\hat{n}) = -i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\chi^*(\hat{n}) = \begin{pmatrix}0 & -1\\ +1 & 0\end{pmatrix}\chi^*(\hat{n}) = \chi(-\hat{n})$$

(2) Cartesian coordinate rotation




$$\begin{pmatrix} V_{x'} \\ V_{z'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V_x \\ V_z \end{pmatrix} = \begin{pmatrix} V_x \cos\theta + V_z \sin\theta \\ -V_x \sin\theta + V_z \cos\theta \end{pmatrix}$$

$$V_q = \begin{pmatrix} \frac{1}{\sqrt{2}}(-V_x - i V_y) \\ V_z \\ \frac{1}{\sqrt{2}}(V_x - i V_y) \end{pmatrix} \quad \begin{matrix} q = +1 \\ q = 0 \\ q = -1 \end{matrix}$$

$$d_{qq'} = \frac{1}{2} \begin{pmatrix} 1+c & -\sqrt{2}s & 1-c \\ \sqrt{2}s & 2c & -\sqrt{2}s \\ 1-c & \sqrt{2}s & 1+c \end{pmatrix} \qquad \begin{matrix} c \equiv \cos\theta \\ s \equiv \sin\theta \end{matrix}$$

$$V'_{+1} = \frac{1}{2}\Big( (1+c)\frac{1}{\sqrt{2}}(-V_x - i V_y) - \sqrt{2}\, s V_z$$
$$(1-c)\frac{1}{\sqrt{2}}(V_x - i V_y) \Big)$$

$$= -\frac{1}{\sqrt{2}}\left(V_x c + s V_z + i V_y\right)$$

Corresponding to

$$V_x' = V_x c + s V_z$$

$$V_y' = V_y$$

$q=0$ term

$V_0' = c V_z - s V_x$ cartesian $V_z' = -s V_x + c V_z$

$q=-1$ term gives

$$V_{-1}' = \frac{1}{\sqrt{2}}\left(c V_x + s V_z - i V_y\right)$$

cartesian $V_x' = c V_x + s V_z$

$V_y' = V_y$

From our similarity transformation matrix

$$S_{iq} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$i \equiv$ cartesian index $= x, y, z$

$q =$ spherical index

we find

$$S_{iq} V_q = S_i = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

and so

$$S V' = \left(S d S^{+}\right) S V$$

is just cartesian rotation:

$$V_i' = \begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix}_{ij} V_j$$

3 a) $\langle \alpha j j | J_1^0 | \alpha' j' j' \rangle = \langle \alpha j \| J_1 \| \alpha' j' \rangle \underbrace{\langle j' j'; 1 0 | j j \rangle}_{\sqrt{j'/(j+1)}}$

also $J_1^0 | \alpha' j' j' \rangle = \hbar j' | \alpha' j' j' \rangle$

giving $\langle \alpha j j | J_1^0 | \alpha' j' j' \rangle = \hbar j \, \delta_{\alpha\alpha'} \delta_{jj'}$

$$= \langle \alpha j \| J_1 \| \alpha' j' \rangle \sqrt{\frac{j}{j+1}}$$

$$\langle \alpha j \| J_1 \| \alpha' j' \rangle = \hbar \delta_{\alpha\alpha'} \delta_{jj'} \sqrt{j(j+1)}$$

b) $\langle \alpha' j m' | \vec{J} \cdot \vec{A} | \alpha j m \rangle =$

$$\langle \alpha' j m' | J^0 A^0 + \tfrac{1}{2} J^- A^+ + \tfrac{1}{2} J^+ A^- | \alpha j m \rangle$$

$$= m\hbar \langle \alpha' j m' | A^0 | \alpha j m \rangle$$

$$+ \frac{\hbar}{2} \sqrt{j(j+1) - m(m-1)} \, \langle \alpha' j m | A^+ | \alpha j m-1 \rangle$$

$$+ \frac{\hbar}{2} \sqrt{j(j+1) - m(m+1)} \, \langle \alpha' j m | A^- | \alpha j m+1 \rangle$$

$$= C \, \langle \alpha' j \| A_1 \| \alpha j \rangle$$

W.E.

Since operator is scalar, C cannot depend on m. By W.E., it cannot depend on $\alpha$, $\alpha'$ or operator A.

So choose $\vec{A} = \vec{J}$,

$$\langle \alpha' j m' | J^2 | \alpha j m \rangle = j(j+1)\hbar^2 \delta_{\alpha\alpha'} \delta_{mm'}$$

$$= C \langle \alpha' j \| J_1 \| \alpha j \rangle = C \, \delta_{\alpha\alpha'} \sqrt{j(j+1)}\, \hbar$$

part a

gives $C = \hbar \, \delta_{mm'} \sqrt{j(j+1)}$

c) $\langle \alpha j m' | A^q | \alpha j m \rangle \underset{WE}{=} \langle \alpha j \| A_1 \| \alpha j \rangle \times \langle j m; 1 q | j m' \rangle$

and $\langle \alpha j \| A_1 \| \alpha j \rangle = \dfrac{1}{\hbar\sqrt{j(j+1)}} \langle \alpha' j m | \vec{J} \cdot \vec{A} | \alpha j m \rangle$

we can get the C.G. coefficient from

$$\langle \alpha j m' | J^q | \alpha j m \rangle = \langle \alpha j \| J_1 \| \alpha j \rangle \times \langle j m; 1 q | j m' \rangle$$

$$= \hbar \sqrt{j(j+1)} \langle j m; 1 q | j m' \rangle$$

Giving $\langle \alpha j m' | A^q | \alpha j m \rangle = \langle \alpha' j m | J \cdot A | \alpha j m \rangle \times \langle \alpha j m' | J^q | \alpha j m \rangle \left( \dfrac{1}{\hbar \sqrt{j(j+1)}} \right)^2$

$$= \frac{1}{\hbar^2 j(j+1)} \langle \alpha' j m | J \cdot A | \alpha j m \rangle \langle \alpha j m' | J^q | j m \rangle$$

4a) Looking at Clebsch Gordon page and recall $Y_\ell^{-m} = (-1)^m (Y_\ell^m)^*$

$$Y_2^{\pm 2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta e^{\pm i 2\phi} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \frac{1}{r^2}(x \pm iy)^2$$

$$Y_2^{\pm 1} = \mp \frac{1}{2}\sqrt{\frac{15}{2\pi}} \sin\theta \cos\theta e^{\pm i\phi} = \mp\frac{1}{2}\sqrt{\frac{15}{2\pi}} \frac{z}{r^2}(x \pm iy)$$

$$Y_2^0 = \frac{1}{4}\sqrt{\frac{15}{\pi}}\left(3\cos^2\theta - 1\right) = \frac{1}{4}\sqrt{\frac{15}{\pi}}\left(\frac{3z^2}{r^2} - 1\right)$$

so $$Y_2^2 + Y_2^{-2} = \frac{1}{2}\sqrt{\frac{15}{2\pi}} \frac{1}{r^2}(x^2 - y^2)$$

$$Y_2^2 - Y_2^{-2} = \frac{i}{2}\sqrt{\frac{15}{2\pi}} \frac{1}{r^2}(xy)$$

$$Y_2^1 - Y_2^{-1} = \sqrt{\frac{15}{2\pi}} \frac{1}{r^2}(xz)$$

$$xy = -2i r^2 \sqrt{\frac{2\pi}{15}}\left(Y_2^2 - Y_2^{-2}\right)$$

$$xz = r^2\sqrt{\frac{2\pi}{15}}\left(Y_2^1 - Y_2^{-1}\right)$$

$$(x^2 - y^2) = 2\sqrt{\frac{2\pi}{15}} r^2 \left(Y_2^2 - Y_2^{-2}\right)$$

$$3z^2 - r^2 = 4\sqrt{\frac{\pi}{15}} Y_2^0$$

so

$$Q = 4\sqrt{\frac{\pi}{15}} \langle \psi | r^2 Y_2^0 | \psi \rangle$$

Apply W.E. $Q = 4\sqrt{\frac{\pi}{15}} \langle \alpha j \| r^2 Y_2 \| \alpha j \rangle \langle jj; 20 | jj \rangle$

Now

$\langle \alpha' j m' | (x^2 - y^2) | \alpha j j \rangle =$

$2\sqrt{\frac{2\pi}{15}} \langle \alpha j m' | r^2 (Y_2^2 + Y_2^{-2}) | \alpha j j \rangle$

$= 2\sqrt{\frac{2\pi}{15}} \langle \alpha j \| r^2 Y_2 \| \alpha j \rangle \left[ \underbrace{\langle j j ; 2 2 | j m' \rangle}_{0} + \langle j j ; 2 -2 | j m' \rangle \right]$

$m' = j + 2 = 3 > \text{max value}$

$= \frac{2\sqrt{\frac{2\pi}{15}}}{4\sqrt{\frac{\pi}{15}}} Q \frac{\langle j j , 2 -2 | j , j-2 \rangle}{\langle j j ; 2 0 | j j \rangle} \delta_{m', j-2}$ $\quad = \frac{1}{\sqrt{2}}$

for $j = 1$, $\langle 1 1 ; 2 0 | 1 1 \rangle = \sqrt{\frac{1}{10}}$

$\langle 1 1 ; 2 -2 | 1 -1 \rangle = \sqrt{\frac{3}{5}}$

$\langle \alpha j , j-2 | (x^2 - y^2) | \alpha j j \rangle = Q \frac{1}{\sqrt{2}} \frac{\sqrt{3/5}}{\sqrt{1/10}} = \sqrt{6}\, Q$