Physics 521 HW 2 Solution Fall 2023

(1)

Uncorrelated random variables

Central limit Theorem

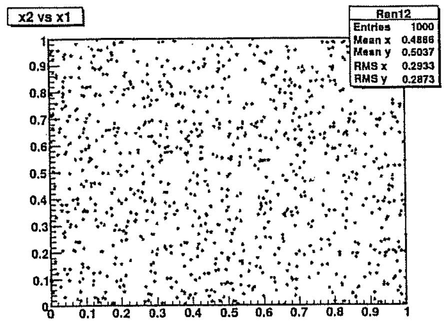


Figure: two uncorrelated, "flat" random variables

Average of N (uncorrelated) random variables:

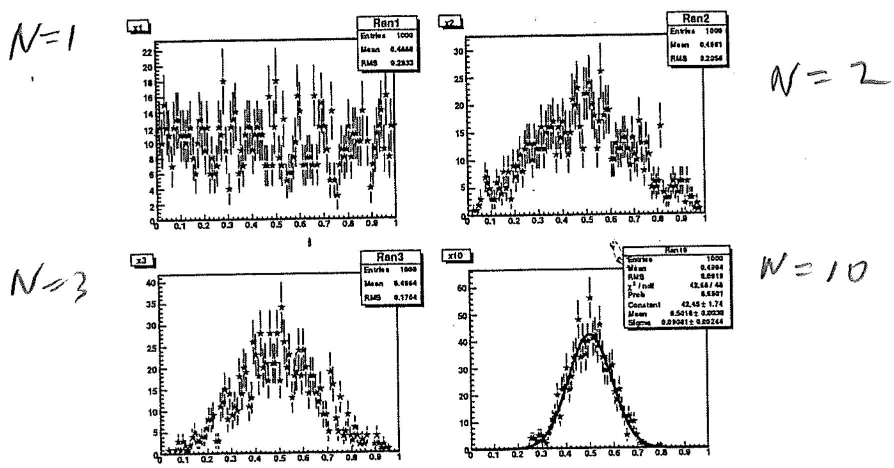


Figure: bottom right (average of 10) is fitted to a Gaussian

From https://pdg.lbl.gov/2023/reviews/rpp2022-rev-probability.pdf

The Gaussian derives its importance in large part from the *central limit theorem*:

If independent random variables $x_1, \ldots, x_n$ are distributed according to *any* p.d.f. with finite mean and variance, then the sum $y = \sum_{i=1}^{n} x_i$ will have a p.d.f. that approaches a Gaussian for large n. If the p.d.f.s of the x_i are not identical, the theorem still holds under somewhat more restrictive conditions. The mean and variance are given by the sums of corresponding terms from the individual x_i. Therefore, the sum of a large number of fluctuations x_i will be distributed as a Gaussian, even if the x_i themselves are not.

#2 $F = \int_0^t \frac{1}{\tau} e^{-t/\tau} dt = 1 - e^{-t/\tau}$

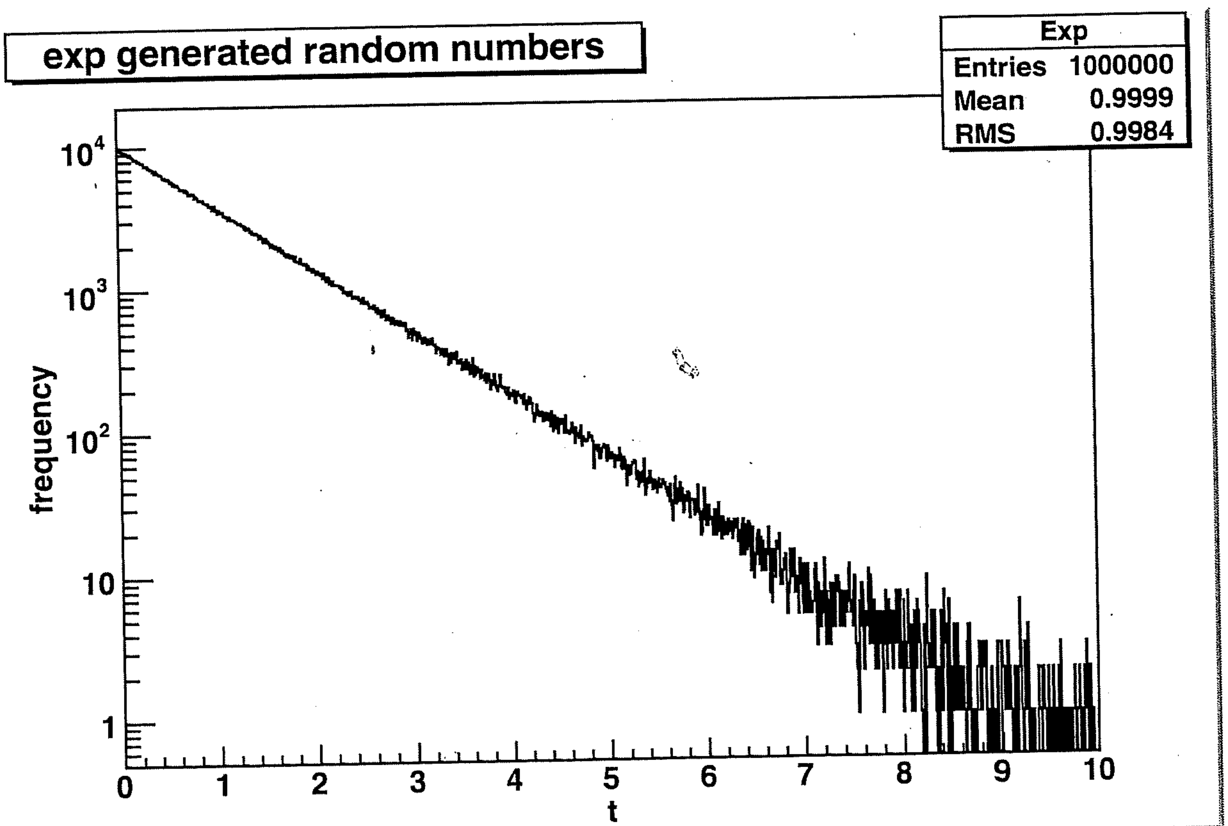
inverting $y = 1 - e^{-t/\tau}$,

$$t = -\tau \ln(1-y)$$

if r is uniform in the interval $(0,1)$ then so is $1-r$. Then

$$\boxed{t = -\tau \ln r}$$

will be exponentially distributed.



#3 To do the Gaussian integral we must square it. Therefore, to generate Gaussian random numbers by the inverse transform method we must generate two at a time. Consider

$$I^2(\rho) = \frac{1}{2\pi\sigma^2}\int_0^{2\pi} d\phi \int_0^{\rho} r\,dr\, e^{-r^2/2\sigma^2}$$

$$= 1 - e^{-\rho^2/2\sigma^2}$$

This is the probability for 2 random Gaussian variables $z_1 = G(x;0,\sigma)$ and $z_2 = G(y;0,\sigma)$ a distance less than ρ from the origin:

$$z_1^2 + z_2^2 < \rho^2$$

To generate z_1, z_2 with this density, <u>invert</u>

$$r = 1 - e^{-\rho^2/2} \qquad (\text{take } \sigma = 1)$$

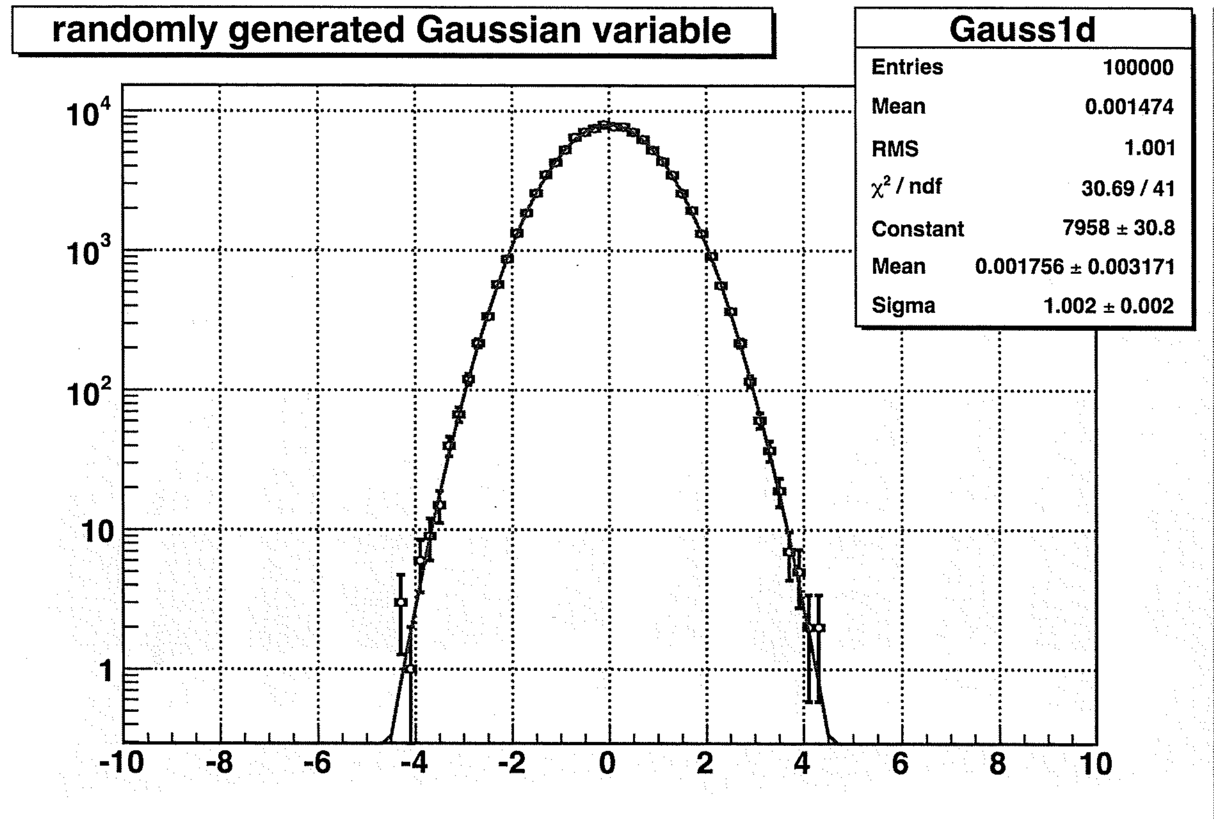
$$\boxed{\rho = \sqrt{-2\ln r}}$$

where r is uniform. Since we want the point $z_1^2 + z_2^2 = \rho^2$ to be uniformly distributed on a circle of this radius, take another uniform random number to get

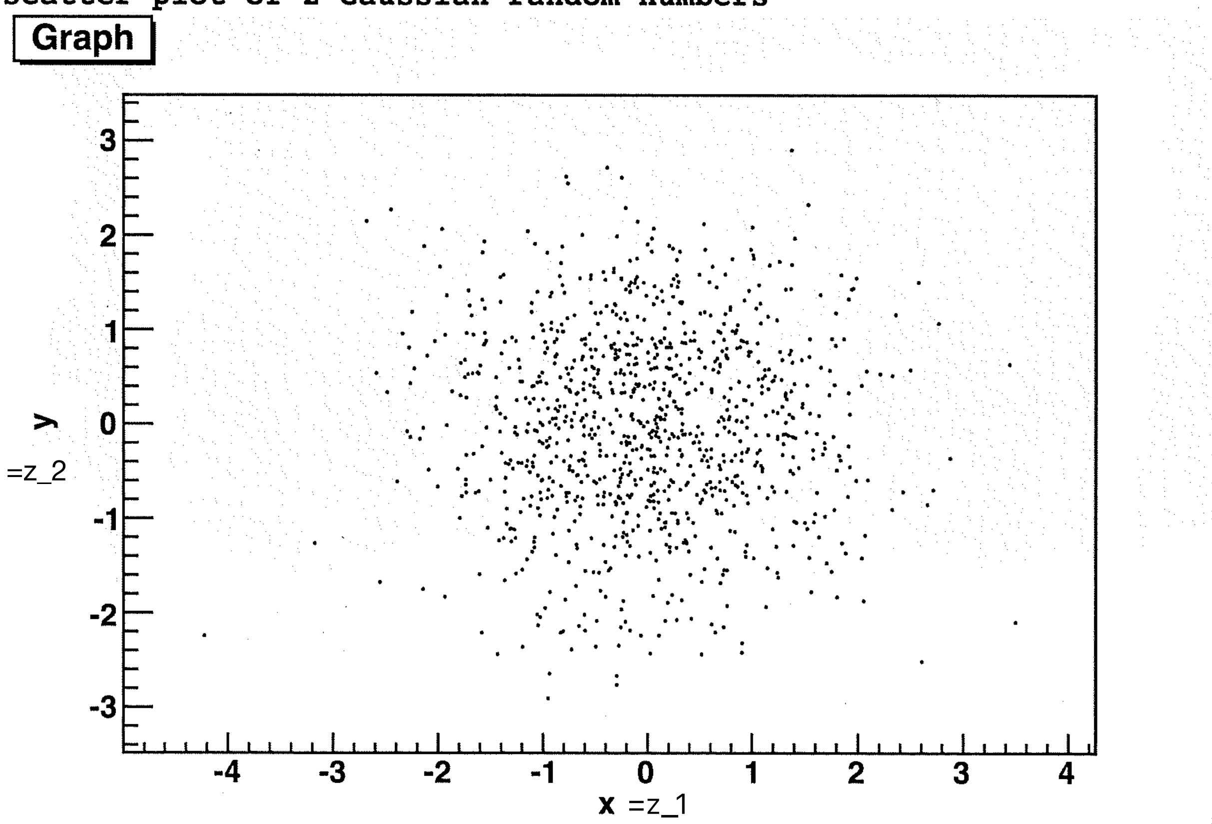
$$z_1 = \sin(2\pi r_2)\sqrt{-2\ln r_1}$$

$$z_2 = \cos(2\pi r_2)\sqrt{-2\ln r_1}$$

log plot of single Gaussian random number generated according to this algorithm. Note the log scale.
The ~~blue~~ line is a likelihood fit to a Gaussian.

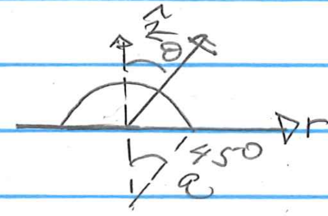


scatter plot of 2 Gaussian random numbers



(4.) Lawn Sprinkler: water drops follow projectile trajectories. Define θ as angle with respect to vertical

$$r(\theta) = \frac{2V_0^2}{g} \cos\theta \sin\theta$$



holes between θ, $\theta + d\theta$

$$dH = \int_0^{2\pi} (a^2 \sin\theta \, d\theta \, d\phi)\, n(\theta)$$

$$= n(\theta)\, 2\pi a^2 \sin\theta \, d\theta$$

n is density of holes in θ

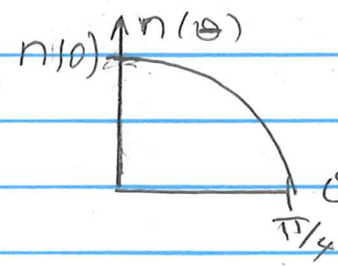
Drops cover area of lawn $r(\theta)$ to $r(\theta + d\theta)$

$$dA = \int_0^{2\pi} r\,dr\,d\phi = 2\pi r\,dr$$

Uniform coverage: $\frac{dH}{dA} = \text{constant}$

$$\frac{2\pi a^2 \sin\theta\, n(\theta)\, d\theta}{2\pi r\, dr} = \text{const.}$$

sketch



$$n(\theta) = (\text{const}) \frac{r}{\sin\theta} \frac{dr}{d\theta}$$

$$\frac{dr}{d\theta} = \frac{2v_0^2}{g}\left(-\sin^2\theta + \cos^2\theta\right)$$

$$n(\theta) = \text{const}\ \frac{cs}{s}(c^2 - s^2) = \text{const} \cos\theta \cos(2\theta)$$

$$= n(0) \cos\theta \cos 2\theta$$

(5)

$$H = \sqrt{p^2+m^2} \qquad (c=1)$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{\sqrt{p^2+m^2}}$$

solve for p

$$\dot{x}^2(p^2+m^2) = p^2 \quad ; \quad p^2 = \frac{m^2\dot{x}^2}{1-\dot{x}^2}$$

$$p = \frac{m\dot{x}}{\sqrt{1-\dot{x}^2}}$$

$$\mathcal{L} = \frac{m\dot{x}^2}{\sqrt{1-\dot{x}^2}} - \left[\frac{m^2\dot{x}^2}{1-\dot{x}^2} + m^2\right]^{1/2}$$

$$= \frac{m\dot{x}^2}{\sqrt{1-\dot{x}^2}} - \frac{1}{\sqrt{1-\dot{x}^2}}\left[m^2\dot{x}^2 + m^2 - m^2\dot{x}^2\right]^{1/2}$$

$$= \frac{m}{\sqrt{1-\dot{x}^2}}(\dot{x}^2-1)$$

$$= -m\sqrt{1-\dot{x}^2}$$

in N.R limit $\dot{x}^2 \ll 1$

$$\mathcal{L}_{NR} = -m\left(1-\tfrac{1}{2}\dot{x}^2\right) = \frac{m}{2}\dot{x}^2 \underbrace{-m}_{\text{irrelevant constant}}$$

⑥ Shanker 2.8.7

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right)$$

$$X(t) = x_0 \cos\omega t + \frac{V_0}{\omega}\sin\omega t \qquad X_0 \equiv X(0) \qquad V_0 = \dot{X}(0)$$

let $X(0) = X_a \qquad X(T) = X_b = X_a \cos\omega T + \frac{V_0}{\omega}\sin\omega T$

$$X(t) = X_a \cos\omega t + \left(\frac{X_b - X_a\cos\omega T}{\sin\omega T}\right)\sin\omega t$$

define $S \equiv \sin\omega T$, $C \equiv \cos\omega T$

$$B \equiv \frac{X_b - X_a\cos\omega T}{\sin\omega T}$$

$$\mathcal{L} = \frac{m\omega^2}{2}\left(X_a \sin\omega t - B\cos\omega t\right)^2 - \frac{m\omega^2}{2}\left(X_a\cos\omega t + B\sin\omega t\right)^2$$

$$= \frac{m\omega^2}{2}\left[(B - X_a^2)(\cos^2\omega t - \sin^2\omega t) - 4B X_a \sin\omega t\cos\omega t\right]$$

$$= \frac{m\omega^2}{2}\left[(B^2 - X_a^2)\cos 2\omega t - 2B X_a \sin 2\omega t\right]$$

then

$$S_{cl} = \int_0^T L\,dt = \frac{m\omega^2}{2}\left(\frac{1}{2\omega}\right)\left(B^2 - X_a^2\right) \sin 2\omega T$$

$$+ \frac{m\omega^2}{2}(2BX_a)\frac{1}{2\omega}\left(\cos^2\omega T - 1\right)$$

$$= \frac{m\omega}{4}\left[\left(B^2 - X_a^2\right)\underbrace{\sin 2\omega T}_{2cs} + 2BX_a\underbrace{\left(\cos^2\omega T - 1\right)}_{-s^2}\right]$$

$$= \frac{m\omega}{2} s\left[\left(B^2 - X_a^2\right)c - 2BX_a s\right]$$

$$2X_a B s = 2X_a\left(X_b - X_a c\right)$$

$$B^2 - X_a^2 = \frac{1}{s^2}\left(X_b^2 - 2X_b X_a c + X_a^2\left(c^2 - s^2\right)\right)$$

$$= \frac{1}{s^2}\left(X_b^2 + X_a^2\right) - \frac{2X_b X_a}{s^2} - 2X_a^2$$

$$S_{cl} = \frac{m\omega}{2} s\left[\frac{c}{s^2}\left(X_b^2 + X_a^2\right) - \frac{2X_b X_a c^2}{s^2} - 2X_a^2 c\right.$$

$$\left. - 2X_b X_a + 2X_a^2 c\right]$$

$$= \frac{m\omega}{2} s\left[\frac{c}{s^2}\left(X_b^2 + X_a^2\right) - \frac{2X_b X_a}{s^2}\right]$$

$$\boxed{S_{cl} = \frac{m\omega}{2\sin\omega T}\left[\cos\omega T\left(X_b^2 + X_a^2\right) - 2X_b X_a\right]}$$

energy is

algebra

$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\omega^2 x^2 = \frac{1}{2} m\omega^2 (A^2 + B^2)$$

with $A = x_a$ and $B = \frac{x_b - x_a c}{s}$

$$E = \frac{1}{2} m\omega^2 \left[x_a^2 + \left(\frac{x_b - x_a c}{s}\right)^2 \right] = \frac{\frac{1}{2} m\omega^2}{s^2} \left[x_a^2 + x_b^2 - 2 x_a x_b c \right]$$

$$\frac{\partial}{\partial t} S_{cl} = -\frac{m\omega^2 c}{2s^2} \left[c(x_a^2 + x_b^2) - 2 x_a x_b \right]$$

$$- \frac{m\omega^2}{s^2} \left[s^2 (x_a^2 + x_b^2) \right]$$

$$= \frac{-m\omega^2}{2s^2} \left[c^2 (x_a^2 + x_b^2) - 2c x_a x_b + s^2 (x_a^2 + x_b^2) \right]$$

$$= \frac{-m\omega^2}{2s^2} \left[x_a^2 + x_b^2 - 2 x_a x_b c \right] = -E$$