Phys 521

HW # 3 Solution

1) Euclidean Vector (spin 1) rotation in spherical basis.

In $\hat{x}, \hat{y}, \hat{z}$ basis, rotation are ($c \equiv \cos\theta, s \equiv \sin\theta$)

$$R(\phi\hat{z}) = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad R(\psi\hat{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

$$R(\theta\hat{y}) = \begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix}$$

The generators $T_i = i\frac{\partial}{\partial\theta_i} R \Big|_{\theta_i = 0}$ are

$$\frac{\hat{J}_z}{\hbar} = \hat{T}_z = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \frac{\hat{J}_x}{\hbar} = \hat{T}_x = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\frac{\hat{J}_y}{\hbar} = \hat{T}_y = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

We can check $[\hat{T}_x, \hat{T}_y] = i\hat{T}_z$

In Q.M. we want a representation where the generator $\hat{T}_z$ is diagonal.

$$\frac{\hat{J}_z}{\hbar}|\mu\rangle = \mu|\mu\rangle$$

diagonalize,

$$\left(\hat{T}_z - \mu\hat{I}\right)|\mu\rangle = 0$$

Follow the usual procedure

$$\det\left(T_z - \mu I\right) = 0 \implies \mu(1-\mu^2) = 0$$

With standard phase convention,

$$\mu = \pm 1 \qquad |\pm 1\rangle \doteq \mp\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}$$

$$|0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Order as $\mu = +1, 0, 1$ similarity transformation to <u>spherical basis</u>

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

<u>Check</u>

$$S^\dagger S = \frac{1}{2}\begin{pmatrix} -1 & +i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & +i & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix} = \hat{I}$$

Then in spherical basis,

$$T_z^s = S^\dagger T_z S = \frac{i}{2}\begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}\underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}}_{\begin{pmatrix} i & 0 & i \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$= \frac{i}{\sqrt{2}}\begin{pmatrix} -2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2i \end{pmatrix} = \underline{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}} \checkmark$$

$$T_y^s = S^\dagger T_y S = \frac{i}{2}\begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}}$$

$$= \underline{\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}} \checkmark$$

$$T_x^s = S^\dagger T_x S = \frac{i}{2}\begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ -i & 0 & -i \end{pmatrix}}$$

$$= \underline{\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} \checkmark$$

We get generators in spherical basis

To get $R^S(\theta\hat{y})$ we can calculate

$$R^S(\theta\hat{y}) = \exp(-i\theta T_y^S)$$

$$= 1 - i\theta T_y^S + \frac{1}{2}(-i\theta)^2 (T_y^S)^2 + \cdots$$

$$= S^\dagger S - i\theta\, S^\dagger T_y S + \frac{1}{2}(-i\theta)^2 \underbrace{(S^\dagger T_y S)^2}$$

$$S^\dagger T_y S = S^\dagger T_y S S^\dagger T_y S$$

$$= S^\dagger \left(1 - i\theta T_y + \frac{1}{2}(-i\theta)^2 T_y^2 + \cdots\right) S$$

$$= S^\dagger R^E(\theta\hat{y})\, S$$

Explicitly,

$$R^S(\theta\hat{y}) = \frac{1}{2}\begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix} \underbrace{\begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}}$$

$$\begin{pmatrix} -c & \sqrt{2}s & c \\ -i & 0 & -i \\ s & \sqrt{2}c & -s \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1+c, & -\sqrt{2}s, & 1-c \\ \sqrt{2}s, & 2c, & -\sqrt{2}s \\ 1-c, & \sqrt{2}s, & 1+c \end{pmatrix} = \begin{pmatrix} \frac{1+c}{2}, & \frac{-s}{\sqrt{2}}, & \frac{1-c}{2} \\ \frac{s}{\sqrt{2}}, & c, & -\frac{s}{\sqrt{2}} \\ \frac{1-c}{2}, & \frac{s}{\sqrt{2}}, & \frac{1+c}{2} \end{pmatrix}$$

<u>spherical rotation about</u> $\hat{y}$

2) $\underline{4,2,1}$ (1) Eigenvalues of $\hat{L}_z$ are $1, 0, -1$

(2) for $|\psi\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\langle \hat{L}_x \rangle = (1\ 0\ 0) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\hat{L}_x^2 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\langle L_x^2 \rangle = \frac{1}{2} (1\ 0\ 0) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} (1\ 0\ 0) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}$$

$$\Delta L_x = \frac{1}{\sqrt{2}}$$

(3) Eigenstates of $\hat{L}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\hat{L}_x \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\det \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right] = 0$$

$$\det \begin{pmatrix} -\lambda & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -\lambda & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -\lambda \end{pmatrix} = -\lambda\left(\lambda^2 - \frac{1}{2}\right) - \frac{1}{\sqrt{2}}\left(\frac{-\lambda}{\sqrt{2}}\right) = 0$$

$$-\lambda\left(\lambda^2 - \frac{1}{2}\right) + \frac{\lambda}{2} = -\lambda^3 + \lambda = 0$$

$$\boxed{\lambda = 1, 0, -1}$$

<u>Eigenvectors</u> $\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix} = \lambda\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}$

$$\frac{1}{\sqrt{2}} C_2 = \lambda C_1$$

$$\frac{1}{\sqrt{2}} C_1 + \frac{1}{\sqrt{2}} C_3 = \lambda C_2$$

$$\frac{1}{\sqrt{2}} C_2 = \lambda C_3$$

and normalization

$$|C_1|^2 + |C_2|^2 + |C_3|^2 = 1$$

$\underline{\lambda = \pm 1}$ $\quad C_2 = \sqrt{2}\,\lambda C_3 = \sqrt{2}\,\lambda C_1$

$$C_1 = C_3 = \frac{1}{2} \;;\; C_2 = \pm\frac{1}{\sqrt{2}}$$

$$|\lambda = \pm 1\rangle \doteq \frac{1}{2}\begin{pmatrix}1\\ \pm\sqrt{2}\\ 1\end{pmatrix}$$

$\underline{\lambda = 0}$ $\quad C_2 = 0 \;,\; C_1 = -C_3 = \frac{1}{\sqrt{2}}$

$$|\lambda = 0\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$$

(5) for State $|-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ measure $\hat{L}_x$

amplitude to measure $\lambda = \pm 1$,

$$\langle \lambda = \pm 1 | -1 \rangle = \frac{1}{2}\left(1, \pm\sqrt{2}, 1\right)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}$$

$\text{Prob} = \left(\frac{1}{2}\right)^2 = 1/4$

amplitude to measure $\lambda = 0$

$$\langle \lambda = 0 | -1 \rangle = \frac{1}{\sqrt{2}}\left(1, 0, -1\right)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}$$

$\text{Prob} = 1/2$

(6) $$|\psi\rangle \doteq \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} = \overbrace{\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

expand in basis of L_z

$$\hat{L}_z \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$L_z^2 \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

measure $\hat{L}_z^2 = 1$ Puts system in state

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ with probability $\frac{1}{4}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ probability $\frac{1}{2}$

measuring L_z, outcomes are

$L_z = 1$, prob $\frac{1}{4}$ $\quad$ $L_z = 0$, prob $\frac{1}{4}$

$L_z = -1$, prob $\frac{1}{2}$

3) Shankar 4.2.2

$\langle p \rangle$ for real ψ $\qquad$ integrate by parts

$$\langle p \rangle = \int_{-\infty}^{\infty} dx\, \psi \frac{\hbar}{i} \frac{\partial \psi}{\partial x} = -\int dx\, \frac{\hbar}{i} \frac{d\psi}{dx} \psi$$

$$= \langle -p \rangle$$

therefore $\langle p \rangle = 0$

4) $\hat{p}_r^c = \frac{\vec{r}}{r} \cdot \vec{p} = \frac{\hbar}{i} \frac{\vec{r} \cdot \vec{\nabla}}{r}$

$\hat{p}_r^{c+} = \vec{p}^+ \cdot \frac{\vec{r}}{r} = \frac{\hbar}{i} \vec{\nabla} \left(\frac{\vec{r}}{r}\right) \qquad (\vec{p}^+ = \vec{p})$

$\neq \hat{p}_r^c$

in cartesian coordinates acting of function $f(\vec{r})$

$$\vec{\nabla} \cdot \left(\frac{\vec{r}}{r} f\right) = \left[\frac{\partial}{\partial x}\frac{x}{r} + \frac{\partial}{\partial y}\frac{y}{r} + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)\right] f$$

$$= \frac{3f}{r} + f x \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + f y \frac{\partial}{\partial y}\left(\frac{1}{r}\right) + f z \frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$+ \frac{x}{r}\frac{\partial}{\partial x} f + \frac{y}{r}\frac{\partial f}{\partial y} + \frac{z}{r}\frac{\partial f}{\partial z}$$

$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{x}{r^3}$ giving

$$= \frac{3f}{r} - \frac{x^2+y^2+z^2}{r^3} f + \frac{1}{r}\vec{r} \cdot \vec{\nabla} f$$

$$= \frac{2f}{r} + \frac{1}{r}\vec{r} \cdot \vec{\nabla} f$$

So

$$\vec{p}_r = \frac{1}{2}\left(\hat{p}_r^c + \hat{p}_r^{c+}\right) = \frac{\hbar}{i}\frac{\vec{r}}{r} \cdot \vec{\nabla} + \frac{1}{r}$$

in spherical coordinates $\frac{\vec{r}}{r} \cdot \nabla = \frac{\partial}{\partial r}$

$$\vec{p}_r f = \frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) f = \frac{\hbar}{i}\left(\frac{1}{r}\frac{\partial}{\partial r} r\right) f$$

so Hermitian $\hat{p}_r = \frac{\hbar}{i}\left(\frac{1}{r}\frac{\partial}{\partial r} r\right)$