HW #3 Problems
Quantum 521

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1. Start with the 3x3 Euclidean rotation matrices (as say given in your mechanics textbook). Calculate the corresponding generators $\hat{J}_x/\hbar, \hat{J}_y/\hbar, \hat{J}_z/\hbar$. Diagonalize $\hat{J}_z/\hbar$ and obtain the similarity transformation $\hat{S}$. Show that in this new basis (called the spherical basis) $\hat{J}_z/\hbar$ is diagonal with values $\hbar, 0, -\hbar$ on the diagonal (this is typically written as $[\hat{J}_z] = diag[\hbar, 0, -\hbar]$). Use the similarity transformation to calculate the matrices $\hat{J}_x/\hbar$ and $\hat{J}_y/\hbar$. Prove that $\hat{S}$ also transforms the rotatioin matrix, and calculate $\hat{S}^\dagger R^E(\theta\hat{y})\hat{S} = R(\theta\hat{y})$.

Exercise 4.2.1 (Very Important). Consider the following operators on a Hilbert space $\mathbb{V}^3(C)$:

$$L_x = \frac{1}{2^{1/2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_y = \frac{1}{2^{1/2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(1) What are the possible values one can obtain if L_z is measured?
(2) Take the state in which $L_z = 1$. In this state what are $\langle L_x \rangle$, $\langle L_x^2 \rangle$, and ΔL_x?
(3) Find the normalized eigenstates and the eigenvalues of L_x in L_z basis.
(4) If the particle is in the state with $L_z = -1$, and L_x is measured, what are the possible outcomes and their probabilities?
(5) Consider the state

$$|\psi\rangle = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2^{1/2} \end{bmatrix}$$

in the L_z basis. If L_z^2 is measured in this state and a result +1 is obtained, what is the state after the measurement? How probable was this result? If L_z is measured, what are the outcomes and respective probabilities?
(6) A particle is in a state for which the probabilities are $P(L_z = 1) = 1/4$, $P(L_z = 0) = 1/2$, and $P(L_z = -1) = 1/4$. Convince yourself that the most general, normalized state with this property is

$$|\psi\rangle = \frac{e^{i\delta_1}}{2}|L_z = 1\rangle + \frac{e^{i\delta_2}}{2^{1/2}}|L_z = 0\rangle + \frac{e^{i\delta_3}}{2}|L_z = -1\rangle$$

It was stated earlier on that if $|\psi\rangle$ is a normalized state then the state $e^{i\theta}|\psi\rangle$ is a physically equivalent normalized state. Does this mean that the factors $e^{i\delta_i}$ multiplying the L_z eigenstates are irrelevant? [Calculate for example $P(L_x = 0)$.]

Figure 1: Shankar 4.2.1

*Exercise 4.2.2.** Show that for a real wave function $\psi(x)$, the expectation value of momentum $\langle P \rangle = 0$. (Hint: Show that the probabilities for the momenta $\pm p$ are equal.) Generalize this result to the case $\psi = c\psi_r$, where ψ_r is real and c an arbitrary (real or complex) constant. (Recall that $|\psi\rangle$ and $\alpha|\psi\rangle$ are physically equivalent.)

Figure 2: Shankar 4.2.2. My hint: I didn't use Shankar's hint.

2. The classical radial momentum is $\frac{\vec{r}}{r}\cdot\vec{p}$. Show that as a quantum operator this is not Hermitian. The operator

$$\hat{p_r} = \frac{1}{2}\left(\frac{\vec{r}}{r}\cdot\vec{p} + \vec{p}\cdot\frac{\vec{r}}{r}\right)$$

is by construction Hermitian. Prove $\hat{p_r} = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}r$.