Phys 521 Fall 2023

HW#4 Solutions

1) $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$ estimate $p \sim \frac{\hbar}{x}$

$$E(x) = \frac{\hbar^2}{2m}\left(\frac{1}{x^2}\right) + \frac{1}{2} m\omega^2 x^2$$

$$\left.\frac{dE}{dx}\right|_{x_m} = 0 = -\frac{\hbar^2}{m x_m^3} + m\omega^2 x_m \quad ; \quad x_m^2 = \frac{\hbar}{m\omega}$$

$$E_m = \frac{\hbar^2}{2m}\left(\frac{m\omega}{\hbar}\right) + \frac{1}{2} m\omega^2\left(\frac{\hbar}{m\omega}\right) = \hbar\omega$$

2) $\Psi(x) = A\left(\frac{L^2}{4} - x^2\right)$ $\quad \psi_0 = \sqrt{\frac{2}{L}} \cos\frac{\pi x}{L}$ normalized to 1

$$\langle \psi_0 | \psi \rangle = \int_{-L/2}^{L/2} \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right) A \left(\frac{L^2}{4} - x^2\right) dx$$

$$y = \frac{\pi x}{L} \quad = \left(\frac{L}{\pi}\right)\sqrt{\frac{2}{L}}\, A \int_{-\pi/2}^{\pi/2} \cos y \left[\frac{L^2}{4} - \frac{L^2}{\pi^2} y^2\right] dy$$

$$\int_{-\pi/2}^{\pi/2} \cos y\, dy = 2 \qquad \int_{-\pi/2}^{\pi/2} y^2 \cos y = \frac{1}{2}\left(\pi^2 - 8\right)$$

$$\langle \psi_0 | \psi \rangle = A\left(\frac{L}{\pi}\right)\sqrt{\frac{2}{L}}\, L^2\left[\frac{1}{2} - \frac{1}{2}\left(1 - \frac{8}{\pi^2}\right)\right] = A\left(\frac{L^3}{\pi}\right)\sqrt{\frac{2}{L}}\left(\frac{4}{\pi^2}\right)$$

$$= A\, L^{5/2} \sqrt{2}\left(\frac{4}{\pi^3}\right)$$

$$P = \left|\langle \psi_0 | \psi \rangle\right|^2$$ so we need A

$$\langle \psi | \psi \rangle = 1 = A^2 \int_{-L/2}^{L/2} \left(\frac{L^2}{4} - x^2\right)^2 dx =$$

$$\frac{L}{\pi} A^2 \int_{-\pi/2}^{\pi/2} \left(\frac{L^2}{4} - \frac{L^2}{\pi^2} y^2\right)^2 dy$$

HW #4-2

$$\frac{1}{A^2} = \frac{L}{\pi}\left(\frac{L^2}{\pi^2}\right)^2 \int_{-\pi/2}^{\pi/2} \left(\frac{\pi^2}{4} - y^2\right)^2 dy$$

$$= \frac{L^5}{\pi^5}\left(\frac{\pi^5}{30}\right) = \frac{L^5}{30}$$

then $P = |\langle \psi_0 | \psi \rangle|^2 = \frac{30}{L^5}\left(\frac{L^5}{\pi^2}\right)2\left(\frac{4}{\pi^2}\right)^2$

$$= 60\left(\frac{4}{\pi^3}\right)^2 = 0.9986$$

3) Free particle Gaussian wave packet time evolution. $E = \hbar^2 k^2/2m$ $\quad \omega(k) = \hbar^2 k^2/2m\hbar$

$$\psi(x,0) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}$$

$$= \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{2\pi}} e^{ikx} \tilde{\psi}(k)$$

$$\psi(x,t) = \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{2\pi}} e^{-i\omega(k)t} e^{ikx} \tilde{\psi}(k)$$

Fourier inversion to get $\tilde{\psi}(k)$

$$\tilde{\psi}(k) = \int dx \frac{1}{\sqrt{2\pi}} e^{-ikx} \psi(x,0)$$

$$= \int dx \frac{1}{\sqrt{2\pi}} e^{-ikx} \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}$$

complete the square in exponential exponent as

$$a\left(x^2 + \frac{b}{a}x\right) = a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2$$

then $\displaystyle\int dx\, e^{-(ax^2+bx)} = e^{b^2/4a} \int dx\, e^{-a\left(x+\frac{b}{2a}\right)^2}$

$$= e^{b^2/4a} \int dy\, e^{-ay^2} = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$$

with $b = ik$

$$\tilde{\psi}(k) = \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{\pi}{a}}\, e^{-k^2/4a}$$

then

$$\psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{\pi}{a}} \int dk \frac{e^{ikx}}{\sqrt{2\pi}} e^{-k^2/4a} e^{-i\omega t}$$

the argument of the exponent is quadratic

$$\arg = -\frac{1}{4a}\left[k^2\left(1 + \frac{2i\hbar a t}{m}\right) + ikx\right]$$

let $\alpha = 1 + 2i\hbar a t/m$

$$\arg = -\frac{\alpha}{4a}\left[k^2 - \frac{4a}{\alpha} ikx\right]$$

$$= -\frac{\alpha}{4a}\left[k - \frac{2ixa}{\alpha}\right]^2 + \frac{\alpha}{4a}\left(-\frac{2ixa}{\alpha}\right)^2$$

blithely shift integration variable to

$$k' = k - \frac{2ixa}{\alpha}$$

$$\psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4}\left(\frac{1}{2\pi}\right)\sqrt{\frac{\pi}{a}}\, e^{-ax^2/\alpha}\int dk'\, e^{-\frac{\alpha}{4a}k'}$$

$$= \left(\frac{2a}{\pi}\right)^{1/4}\frac{e^{-ax^2/\alpha}}{\sqrt{\alpha}}$$

prob. density

$$|\psi(x,t)|^2 = \sqrt{\frac{2a}{\pi}}\,\frac{e^{-ax^2\left(\frac{1}{\alpha}+\frac{1}{\alpha^*}\right)}}{|\alpha|}$$

u write $\alpha = 1 + i\beta \qquad \beta \equiv \frac{2\hbar a t}{m}$

$$|\alpha| = \sqrt{1+\beta^2}$$

$$\frac{1}{\alpha} + \frac{1}{\alpha^*} = \frac{2}{1+\beta^2}$$

$$|\psi(x,t)|^2 = \sqrt{\frac{2a}{\pi}}\,\frac{1}{\sqrt{1+\beta^2}}\exp\left(\frac{-2ax^2}{1+\beta^2}\right)$$

rewrite with $\gamma \equiv \left(\frac{a}{1+\beta^2}\right)^{1/2} = \left[\frac{a}{1+\left(\frac{2\hbar a t}{m}\right)^2}\right]^{1/2}$

$$|\psi(x,t)|^2 = \sqrt{\frac{2}{\pi}}\,\gamma\, e^{-2\gamma^2 x^2}$$

a properly normalized Gaussian with width

$$\sigma(t) = \frac{1}{2\gamma} = \frac{1}{2\sqrt{a}}\left[1+\left(\frac{2\hbar a t}{m}\right)^2\right]^{1/2}$$

check at $t=0$, $\gamma^2 = a$.

wave packet expands with characteristic time

$$\tau = \left(\frac{m}{2\hbar a}\right)$$

$$\sigma(t) = \frac{1}{2\sqrt{a}}\left[1+\left(\frac{t}{\tau}\right)^2\right]^{1/2}$$

note $\frac{1}{\sqrt{a}}$ has dimension of length

Uncertainty product.

$\psi(x,t) = \psi(-x,t)$ so $\langle x\rangle = 0$

$$\langle p\rangle = \int dx\, \psi^* \frac{\hbar}{i}\frac{\partial}{\partial x}\psi\, dx$$

$\frac{\partial}{\partial x}\psi = -\frac{2a}{\alpha} x\psi$ so $\langle p\rangle \propto \langle x\rangle = 0$

$$\langle x^2\rangle = \int \sqrt{\frac{2}{\pi}}\,\gamma\, e^{-2\gamma^2 x^2} x^2\, dx = \sigma^2(t) \text{ (Gaussian)}$$

$$\langle p^2 \rangle = \int dx\, \psi^* \; -\hbar \frac{\partial^2}{\partial x^2} \psi$$

$$\frac{\partial^2}{\partial x^2}\psi = -\frac{2a}{\alpha}\psi + \frac{4a^2}{\alpha^2} x^2 \psi$$

$$\langle p^2 \rangle = \hbar^2 \left(\frac{2a}{\alpha}\right) - \hbar^2 \frac{4a^2 \sigma^2}{\alpha^2}$$

write again $\alpha = 1 + i\beta$

$$\alpha^{-1} = \frac{1-i\beta}{1+\beta^2} \; ; \; \alpha^{-2} = \frac{1-2i\beta-\beta^2}{(1+\beta^2)^2}$$

$$\sigma^2 = \frac{1}{4a}(1+\beta^2)$$

then

$$\langle p^2 \rangle = \hbar^2 \left(\frac{2a}{1+\beta^2}\right)(1-i\beta) - \hbar^2 \frac{4a^2(1+\beta^2)}{4a} \frac{(1-2i\beta-\beta^2)}{(1+\beta^2)^2}$$

$$= \frac{\hbar^2 a}{1+\beta^2}\left[2 - 1 + \beta^2\right] = a\hbar^2$$ (thankfully real)

$$\Delta x^2 \Delta p^2 = \sigma^2 a^2 \hbar^2 = \frac{1}{4}\left(1 + \left(\frac{t}{\tau}\right)^2\right)\hbar^2$$

$$\Delta x \Delta p = \frac{\hbar}{2}\left[1 + \left(\frac{t}{\tau}\right)^2\right]^{1/2} \geq \frac{\hbar}{2}$$

grows with characteristic time $\tau = \frac{m}{2\hbar a}$

#4) $$P = e^{-t/\tau}$$

$$\frac{dP}{dt} = -\frac{1}{\tau} e^{-t/\tau}$$

from formula in lecture #6

$$\Delta E \left(P - P^2 \right)^{1/2} \geq \frac{\hbar}{2} \left| \frac{dP}{dt} \right|$$

$$(\Delta E)^2 \left(e^{-t/\tau} - e^{-2t/\tau} \right) \geq \left(\frac{\hbar}{2} \frac{1}{\tau} \right)^2 e^{-2t/\tau}$$

$$\left(e^{t/\tau} - 1 \right) \geq \left(\frac{\hbar}{2\tau \Delta E} \right)^2$$

$$t \geq \tau \ln \left(1 + \left(\frac{\hbar}{2\tau \Delta E} \right)^2 \right)$$

Lab measurement of lifetime is done on a statistical ensemble of identically prepared states.