Phys 521 Fall 2023

Hw #5 Solutions

1) $\langle \sigma_k \rangle = \mathrm{tr}(\rho\, \sigma_k)$

$$= \mathrm{tr}\left[\tfrac{1}{2}(I + \vec{P}\cdot\vec{\sigma})\sigma_k\right]$$

$$= \frac{1}{2} P_i \underbrace{\mathrm{tr}(\sigma_i \sigma_k)}_{2\delta_{ki}} = P_k$$

2 a) knowing $\langle \sigma_i \rangle = P_i \frac{\hbar}{2}$ for $i = 1,2,3$
determine

$$[\rho] \doteq_z \frac{1}{2}\begin{bmatrix} 1+P_z, & P_x - iP_y \\ P_x + iP_y, & 1+P_z \end{bmatrix}$$

b) An arbitrary pure state is given by

$$|\psi\rangle = \cos\theta/2\,|+z\rangle + e^{i\phi}\sin\theta/2\,|-z\rangle$$

$$\rho = |\psi\rangle\langle\psi| \doteq_z \begin{pmatrix} c^2 & e^{i\phi} cs \\ e^{-i\phi} cs & s^2 \end{pmatrix} \qquad \begin{aligned} c &= \cos\theta/2 \\ s &= \sin\theta/2 \end{aligned}$$

we find $\frac{1}{2}(1+P_z) = \cos^2\theta/2 = \frac{1}{2}(1+\cos\theta)$

$\frac{1}{2}(1-P_z) = \sin^2\theta/2 = \frac{1}{2}(1-\cos\theta)$

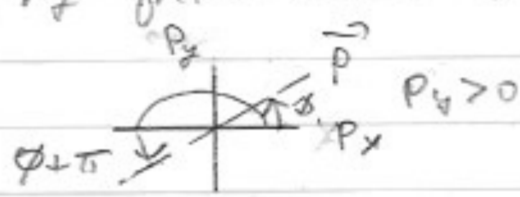
So $\cos\theta = P_z$

and $P_x \pm i P_y = \underbrace{2\cos\theta/2 \sin\theta/2}_{\sin\theta} (\cos\phi \pm i \sin\phi)$

$P_x = \sin\theta \cos\phi$
$P_y = \sin\theta \sin\phi$

knowing P_x determines ϕ as $\phi = \cos^{-1}\left(\frac{P_x}{\sqrt{1-P_z^2}}\right)$

up to ϕ or $\phi + \pi$. Specifying the sign of P_y fixes this π term.



3) Given the pure density matrix $\hat{\rho}(0)$ at $t=0$, at time t,

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| = \hat{U}(t,0)|\psi(0)\rangle\langle\psi(0)|\hat{U}^\dagger(t,0)$$

We have for all t, $\hat{\rho}^2 = 1$ so system remains pure:

$$\hat{\rho}^2(t) = \hat{U}(t,0)|\psi(0)\rangle\langle\psi(0)|\underbrace{\hat{U}^\dagger(t,0)\hat{U}(t,0)}_{I}|\psi(0)\rangle\langle\psi(0)|\hat{U}^\dagger(t)$$

$$= \hat{U}(t,0)|\psi(0)\rangle\langle\psi(0)|\hat{U}^\dagger(t) = \hat{\rho}(t)$$

4) $\frac{d}{dt}\langle S_k \rangle = tr\left(\frac{\partial \rho}{\partial t} S_k\right) \qquad S_k = \frac{\hbar}{2}\sigma_k$

$$\hat{H} = -\vec{\mu}\cdot\vec{B} = -\left(-\frac{eg\hbar}{2mc}\right)\frac{\vec{S}\cdot\vec{B}}{\hbar}$$

define $\hbar\vec{\omega} = \frac{2g\hbar}{2mc}\vec{B}$, $\hat{H} = \vec{\omega}\cdot\vec{S} = \frac{\hbar}{2}\vec{\omega}\cdot\vec{\sigma}$

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H,\rho] = \frac{1}{i\hbar}\left(\frac{\hbar}{2}\right)\frac{1}{2}\left[\vec{\sigma}\cdot\vec{\omega}, \vec{P}\cdot\vec{\sigma}\right]$$

$$= \frac{1}{i\hbar}\left(\frac{\hbar}{2}\right)\frac{1}{2}\,\omega_i P_j \underbrace{[\sigma_i,\sigma_j]}_{2i\varepsilon_{ijk}\sigma_k}$$

$$= \frac{1}{2}\varepsilon_{ijk}\,\omega_i P_j \sigma_k = \frac{1}{2}(\vec{\omega}\times\vec{P})\cdot\vec{\sigma}$$

Then $\frac{d}{dt}\langle S_k\rangle = tr\left(\frac{\partial\rho}{\partial t} S_k\right)$

$$= \frac{1}{2}\left(\frac{\hbar}{2}\right)\varepsilon_{ij\ell}\,\omega_i P_j\, \underbrace{tr(\sigma_\ell \sigma_k)}_{2\delta_{\ell k}} = \frac{\hbar}{2}\varepsilon_{ijk}\,\omega_i P_j$$

$$= \frac{\hbar}{2}\left(\vec{\omega}\times\vec{P}\right)_k$$

giving $\frac{d}{dt}\langle\vec{S}\rangle = \vec{\omega}\times\langle\vec{S}\rangle$

classically with charge $q = -e$

$$\frac{d\vec{S}}{dt} = \tau = \vec{\mu}\times\vec{B} = -\vec{S}\times\vec{\omega} = \vec{\omega}\times\vec{S}$$