Phys 521 Fall 2023

HW#6 - Solutions

1) $E = \frac{p_z^2}{2m} + mgz$ estimate $p \sim \frac{\hbar}{z}$ for ground state

$$E(z) = \frac{\hbar^2}{2m}\left(\frac{1}{z^2}\right) + mgz$$

$$\left.\frac{dE}{dz}\right|_{z_m} = 0 \quad \text{gives} \quad z_m^3 = \frac{\hbar^2}{m_n g} = \frac{(\hbar c)^2}{m_n c^2 (m_n g)}$$

$$z_m^3 = \frac{(0.2\,\text{eV}\,\mu\text{m})^2}{10^{9}\,\text{eV} \times 10^{-13}\,\text{eV}/\mu\text{m}} = 400\,\mu\text{m}^3$$

$z_m \doteq 7\,\mu\text{m}$ see "bouncing neutrons" link

2) Translations in momentum space,

$$\hat{W}(p_0) = e^{i p_0 \hat{x}/\hbar}$$

$$[\hat{p}, \hat{W}] = i\hbar \frac{\partial}{\partial \hat{x}} \hat{W}(p_0) = p_0 \hat{W}(p_0)$$

so $\hat{W}(p_0)|p\rangle = |p + p_0\rangle$

For infinitesimal $\hat{W}(\varepsilon) = 1 + i\varepsilon \hat{x}/\hbar$

$$\hat{W}(\varepsilon)|\psi\rangle = (1 + i\varepsilon\hat{x}/\hbar)|\psi\rangle = \int dp\, \hat{W}(\varepsilon)|p\rangle\langle p|\psi\rangle$$

$$= \int dp\, |p+\varepsilon\rangle\langle p|\psi\rangle = \int dp'\, |p'\rangle\langle p'-\varepsilon|\psi\rangle$$

expand $\langle p'-\varepsilon|\psi\rangle = \tilde{\psi}(p'-\varepsilon) = \tilde{\psi}(p') - \varepsilon\frac{\partial}{\partial p'}\tilde{\psi}(p')$

then act on left with bra $\langle p|$

$$\langle p|(1+i\varepsilon\hat{x}/\hbar)|\psi\rangle = \tilde{\psi}(p) + \frac{i\varepsilon}{\hbar}\langle p|\hat{x}|\psi\rangle$$

$$\int dp' \underbrace{\langle p|p'\rangle}_{\delta(p-p')}\left(\tilde{\psi}(p') - \varepsilon\frac{\partial}{\partial p'}\tilde{\psi}(p')\right)$$

$$= \psi(p) - \varepsilon\frac{\partial}{\partial p}\tilde{\psi}(p)$$

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle$$

or as a matrix in p-basis, put $|\psi\rangle = |p'\rangle$

$$\langle p|\hat{x}|p'\rangle = i\hbar\frac{\partial}{\partial p}\delta(p-p') = i\hbar\,\delta(p-p')\frac{\partial}{\partial p'}$$

conclude in p basis $\hat{x} \underset{p}{\doteq} i\hbar\frac{\partial}{\partial p}$

check

$$\left[i\hbar\frac{\partial}{\partial p}, p\right]\tilde{f}(p) = i\hbar\frac{\partial}{\partial p}(p\tilde{f}) - p\,i\hbar\frac{\partial\tilde{f}}{\partial p} = i\hbar\tilde{f}$$

so $\left[i\hbar\frac{\partial}{\partial p}, p\right] = i\hbar$

3) From lecture #8, S-matrix for

$$V(x) = -\frac{\hbar^2}{2m}\frac{1}{b}\delta(x) \qquad \dim[b] = \text{length}$$

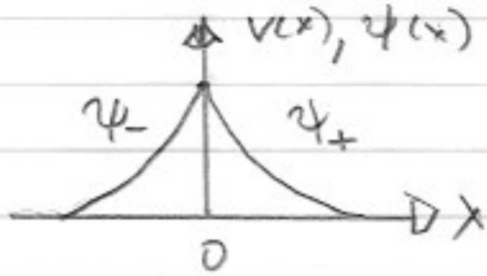
$$[S] = \frac{1}{2kb-i}\begin{pmatrix} i, & 2kb \\ 2kb, & i \end{pmatrix}$$

simple pole for complex $k = \frac{i}{2b} \equiv iq$

bound state problem:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi - \frac{\hbar^2}{2m}\frac{1}{b}\delta(x)\,\psi = E\psi$$

$$\psi'' + \frac{\delta(x)}{b}\psi = q^2\psi \qquad q \equiv \frac{\sqrt{2m|E|}}{\hbar}$$



ψ has discontinuity in slope at $x=0$

$$\psi_\pm = A e^{\mp q x} \qquad \psi_+(0) = \psi_-(0)$$

discontinuity of derivative from integrating over δ-function

$$\int_{-\varepsilon}^{+\varepsilon} dx\, \psi'' + \frac{1}{b}\psi(0) = q^2 \psi(0) \int_{-\varepsilon}^{+\varepsilon} dx$$

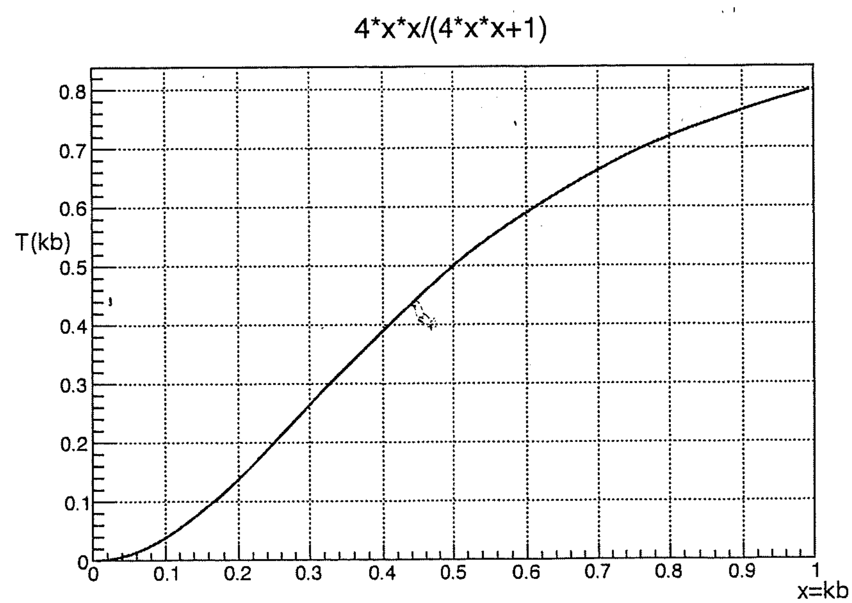
lim $\varepsilon \to 0$

$$\left.\frac{d\psi_+}{dx}\right|_0 - \left.\frac{d\psi_-}{dx}\right|_0 = -\frac{1}{b} A$$

$$A\left[-q \;\; -q\right] = -\frac{1}{b} A \quad \Rightarrow q = \frac{1}{2b}$$

which is pole of S-matrix.

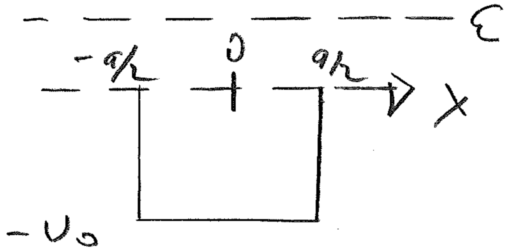
Plotting transmission probability $T = \dfrac{4x^2}{4x^2+1}$

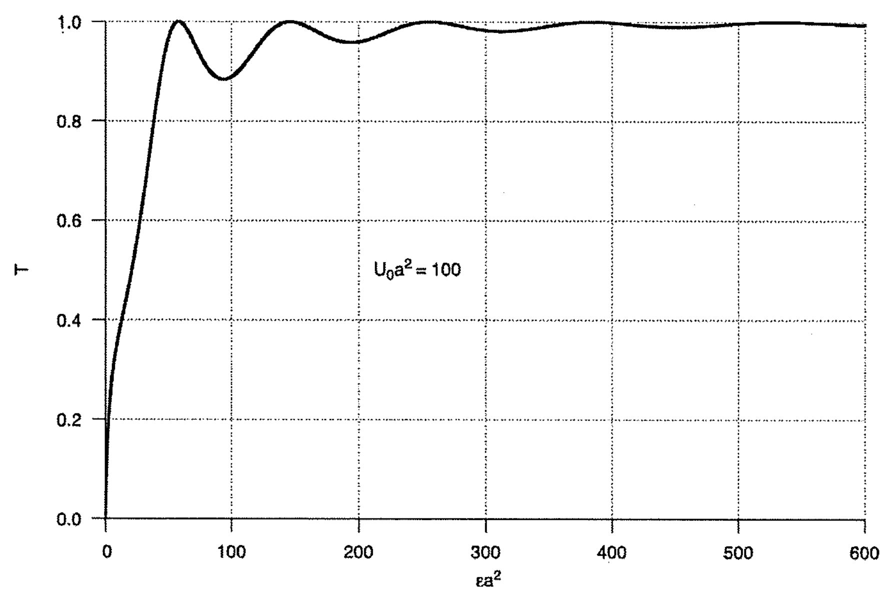


$$T'' = \frac{8}{(1+4x^2)^3}\left[1 - 12x^2\right]$$ inflection point of curve is $\frac{1}{\sqrt{12}} \cong 0.3$

actually not at $x = 1/2$

resonance in scattering example:

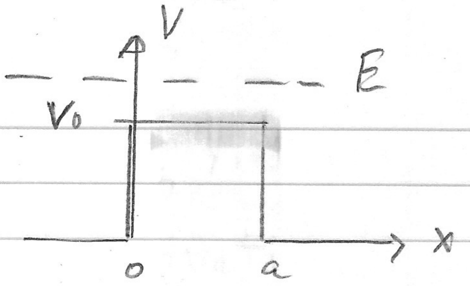




Transmission coefficient for the potential well of Figure 6.5 with $U_0a^2 = 100$, plotted as a function of εa^2

From Commins

4)



$k = \sqrt{2mE}/\hbar$

$k' = \sqrt{2m(E-V_0)}/\hbar$

$$\psi(x) = \begin{cases} e^{ikx} + Be^{-ikx} & x<a \\ Fe^{ik'x} + Ge^{-ik'x} & 0<x<a \\ Ce^{ikx} & x>a \end{cases}$$

$1 + B = F + G \qquad Fe^{ik'a} + Ge^{-ik'a} = Ce^{ika}$

$\frac{k}{k'}(1-B) = F - G \qquad Fe^{ik'a} - Ge^{-ik'a} = \frac{k}{k'}Ce^{ika}$

$r \equiv k/k'$

$1 + r + B(1-r) = 2F$

$1 - r + B(1+r) = 2G$

$(1+r)Ce^{ika} = 2Fe^{+ik'a}$

$(1-r)Ce^{ika} = 2Ge^{-ik'a}$

$e^{-ik'a}(1+r)\, Ce^{ika} = 2F = 1 + r + B(1-r)$

$e^{+ik'a}(1-r)\, Ce^{ika} = 2G = 1 - r + B(1+r)$

$(1+r)^2 e^{-ik'a} Ce^{ika} = (1+r)^2 + B(1-r^2)$

$(1-r)^2 e^{+ik'a} Ce^{ika} = (1-r)^2 + B(1-r^2)$

Subtract

$$C e^{ika}\left[(1+r)^2 e^{-ik'a} - (1-r^2)\, e^{ik'a}\right] = 4r$$

$$C e^{ika}\left[(1+2r+r^2) e^{-ik'a} - (1-2r+r^2)\, e^{ik'a}\right] = 4r$$

$$C e^{ika}\left[(1+2r^2)(2i)\sin k'a + 4r\cos k'a\right] = 4r$$

square

$$|C|^2\left[4(1+r^2)^2\sin^2 k'a + (4r^2)^2\cos^2 k'a\right] = (4r)^2$$

$$T = |C|^2 = \left[\cos^2 k'a + \frac{(1+r^2)^2}{4r^2}\sin^2 k'a\right]^{-1}$$

with $\left(\frac{1+r^2}{2r}\right)^2 = \left(\frac{k^2+k'^2}{2k'k}\right)^2$

For barrier penetration, analytically continue with

$k' = iq \qquad q = \sqrt{2m(V_0-E)}/\hbar$

$$\cosh^2 qa = 1 + \sinh^2 qa$$

$$\frac{(k^2-q^2)^2}{4q^2k^2} + 1 = \frac{(k^2+q^2)^2}{4q^2k^2}$$

$$T = \left[1 + \left(\frac{k^2+q^2}{2qk}\right)^2 \sinh^2 qa\right]^{-1}$$