Phys 521 Fall 2023

HW # 7 Solutions

① Double δ-well $V(x) = -V_0\left[\delta(x+a) + \delta(x-a)\right]$

bound state $E<0$ $q = \sqrt{-2mE}/\hbar$

$$\psi = e^{\pm qx}$$

$$\psi^+(x) = \begin{cases} Ae^{+qx} & x<-a \\ B\cosh qx & -a<x<a \\ A e^{-qx} & x>a \end{cases}$$ symmetric

$$\psi^-(x) \begin{cases} -Ae^{qx} & x<-a \\ B\sinh qx & -a<x<a \\ Ae^{-qx} & x>a \end{cases}$$ antisymmetric

Ground state is the symmetric solution
Apply boundary conditions:

symmetric
$$Ae^{-qa} = B\cosh qa$$
$$Ae^{-qa} - Bq\sinh qa = -\frac{A}{b}e^{-qa}$$

$b \equiv \dfrac{\hbar^2}{2mV_0}$ length

$$\tanh q_+ a = \frac{1}{q_+ b} - 1$$

Similarly, antisymmetric

$$\coth q_- a = \frac{1}{q_- b} - 1$$

Only ground state exists if

$$\frac{1}{qb} - 1 > 1 \quad ; \quad q < \frac{1}{2b}$$

$$q^2 = \frac{-2mE}{\hbar^2} < \frac{1}{4}\left(\frac{2mV_0}{\hbar^2}\right)^2$$

$$E > -\frac{1}{2}\,\frac{mV_0^2}{\hbar^2}$$

limit $mV_0 a \gg \hbar^2$ $\qquad a \gg b$

let $y = qa$.

$$\tanh y = \frac{a}{b}\frac{1}{y} - 1 \approx \frac{a}{b}\frac{1}{y}$$

$$\coth y = \frac{a}{b}\frac{1}{y} - 1 \approx \frac{a}{b}\frac{1}{y}$$

In this limit $\tanh y \approx \coth y \approx 1$ and

$$q = \frac{1}{b} \qquad q^2 = \frac{-2mE}{\hbar^2} = \frac{1}{b^2}$$

$$E = -\frac{\hbar^2}{2m}\left(\frac{1}{b^2}\right) = -\frac{\hbar^2}{2m}\left(\frac{2mV_0}{\hbar^2}\right)^2 = -\frac{2mV_0^2}{\hbar^2}$$

this is twice the energy of the single δ-function potential, as the particle is bound by two δ-functions.

As $a \to \infty$, the symmetric & antisymmetric probability density functions become the same, so the energies become the same.

2) Scattering off of a potential non-zero in interval $0<x<a$ is described by the S-matrix

$$\psi_{in} = Ae^{ikx} + Be^{-ikx} \qquad x<a$$

$$\psi_{scat} = Fe^{ikx} + Ge^{-ikx} \qquad x>a$$

$$\begin{pmatrix} B \\ F \end{pmatrix} = [S] \begin{pmatrix} A \\ G \end{pmatrix}$$

T_L incident from left $(G=0)$

$$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{12} & S_{21} \\ S_{12} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$F = S_{12} A \qquad T_L = \left|\frac{F}{A}\right|^2 = |S_{12}|^2$$

T_R incident from right $(A=0)$

$$T_R = |S_{21}|^2$$

The S-matrix is unitary,

$$S_{21}^* = S_{12}$$

$$|S_{21}|^2 = |S_{12}|^2$$

so $\quad T_R = T_L$

3) In terms of the dimensionless variables

$$\hat{y} = \hat{x}\sqrt{\frac{m\omega}{\hbar}} \qquad \hat{q} = \hat{p}/\sqrt{m\omega\hbar}$$

$$\Delta x = \sqrt{\frac{\hbar}{m\omega}}\,\Delta y \qquad \Delta p = \sqrt{\hbar m\omega}\,\Delta q$$

$$\Delta x \Delta p = \hbar\,\Delta y \Delta q$$

$$\hat{y} = \frac{1}{\sqrt{2}}(a^+ + a) \qquad \hat{q} = \frac{i}{\sqrt{2}}(a^+ - a)$$

then $\langle n|\hat{y}|n\rangle = 0$ & $\langle n|\hat{q}|n\rangle = 0$

$$\hat{y}^2 = \tfrac{1}{2}(a^+ + a)(a^+ + a) = \tfrac{1}{2}(a^{+2} + a^2 + \underbrace{a^+a + aa^+}_{2\hat{n}+1})$$

$$= \tfrac{1}{2}(a^{+2} + a^2) + \hat{n} + \tfrac{1}{2}$$

so $\langle n|\hat{y}^2|n\rangle = n + \frac{1}{2}$

similarly $\hat{q}^2 = -\frac{1}{2}(a^{+2} + a^2) + \hat{n} + \frac{1}{2}$

giving $\Delta y \Delta q = n + \frac{1}{2}$

$$\Delta x \Delta p = \hbar\left(n + \tfrac{1}{2}\right)$$

4) time evolution

$$\frac{d}{dt}\langle\hat{x}\rangle = \frac{i}{\hbar}\langle[H,\hat{x}]\rangle$$

$$H = \hbar\omega\left(a^{+}a + \frac{1}{2}\right)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(a^{+} + a\right)$$

$$[\hat{H},\hat{x}] = \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left[a^{+}a, a^{+}+a\right]$$

$$= \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left(\underbrace{[a^{+}a, a^{+}]}_{a^{+}} + \underbrace{[a^{+}a, a]}_{-a}\right)$$

$$= \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left(a^{+}-a\right) = \frac{\hbar}{im}\hat{p}$$

similarly, $[\hat{H},\hat{p}] = i\hbar m\omega^2\hat{x}$

giving $\frac{d}{dt}\langle x\rangle = \frac{1}{m}\langle p\rangle$

$$\frac{d}{dt}\langle p\rangle = -m\omega^2\langle x\rangle$$

5) parity operator for the harmonic oscillator is

$\hat{P}|n\rangle = \pm|n\rangle$ $\quad$ $+$ n even $\quad 0, 2, 4, \dots$

$\quad$ $-$ n odd $\quad 1, 3, 5, \dots$

$e^{i\pi} = -1$ so $(e^{i\pi})^n = \begin{cases} +1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$

it follows that

$$\hat{P} = \exp(i\pi\hat{n})$$

given $\hat{P}|n\rangle = e^{i\pi n}|n\rangle$

6) This adds to the Hamiltonian

$$V' = -q\mathcal{E}\hat{x} = -\hbar\omega\lambda\hat{y} \qquad \lambda \equiv \frac{q\mathcal{E}}{\hbar\omega}\sqrt{\frac{\hbar}{m\omega}}$$

then $\hat{H} = \frac{\hbar\omega}{2}(\hat{q}^2 + \hat{y}^2) - \hbar\omega\lambda\hat{y}$

$$= \frac{\hbar\omega}{2}(\hat{q}^2 + \hat{y}^2 - 2\lambda\hat{y})$$

let $\hat{y}' = \hat{y} - \lambda$

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{q}^2 + \hat{y}'^2 - \lambda^2)$$

effect is just to shift the energies

$$E_n = \hbar\omega\left(n + \frac{1}{2} - \frac{1}{2}\lambda^2\right)$$

7) In the $|I\rangle, |II\rangle$ basis the Hamiltonian is diagonal

$$[H]_{I,II} = \begin{pmatrix} E_0 - A & 0 \\ 0 & E_0 + A \end{pmatrix}$$

states $|I\rangle, |II\rangle$ have simple time dependence

at $t=0$, $\quad |\psi(0)\rangle = |1\rangle \underset{I,II}{\doteq} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

state $\quad |2\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-iE_0 t/\hbar} \begin{pmatrix} e^{iAt/\hbar} \\ e^{-iAt/\hbar} \end{pmatrix}$$

$$\langle 2|\psi(t)\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 e^{-iE_0 t/\hbar} (1, -1) \begin{pmatrix} e^{iAt/\hbar} \\ e^{-iAt/\hbar} \end{pmatrix}$$

$$= i e^{-iE_0 t/\hbar} \sin\left(\frac{At}{\hbar}\right)$$

$$P_{1\to 2}(t) = \sin^2(At/\hbar)$$