Hw #9 Solutions

1) $$f(\theta_i) = \lim_{n\to\infty}\left(1 - \frac{i\theta_i\hat{T}_i}{n}\right)^n$$

$$\frac{\partial f}{\partial \theta_j} = \lim_{n\to\infty} n\left(1 - \frac{i\theta_i\hat{T}_i}{n}\right)^{n-1}\left(-i\frac{\hat{T}_j}{n}\right)$$

$$= -iT_j \lim_{n\to\infty} \frac{\left(1 - \frac{i\theta_i\hat{T}_i}{n}\right)^n}{1 - \frac{\theta_i\hat{T}_i}{n}}$$

$$= -iT_j \lim_{n\to\infty}\left(1 + \frac{i\theta_i\hat{T}_i}{n}\right)^n = -iT_j f$$

therefore, it is the exponential,

$$f(\theta_i) = \exp(-i\theta_i\hat{T}_i)$$

2) $$[J_jJ_j, J_i] = J_jJ_jJ_i - J_iJ_jJ_j$$

$$= J_j(i\varepsilon_{jik}J_k) - i\varepsilon_{ijk}J_kJ_j$$

$$= -i\varepsilon_{ijk}J_jJ_k - i\varepsilon_{ijk}J_kJ_j$$ switch dummy indicies

$$= -i\varepsilon_{ijk}J_jJ_k - i\varepsilon_{ikj}J_jJ_k$$

$$= -iJ_jJ_k(\varepsilon_{ijk} + \varepsilon_{ikj}) = 0$$

3) $J_+|1,-1\rangle = \hbar\sqrt{2-(-1)-(-1+1)}\,|1,0\rangle$

$= \hbar\sqrt{2}\,|1,0\rangle$

$J_+|1,0\rangle = \hbar\sqrt{2}\,|1,1\rangle$

$$J_+ = \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_- = J_+^{\dagger} = \sqrt{2}\hbar\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_x = \frac{1}{2}\left(J_+ + J_-\right) = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_y = \frac{1}{2i}\left(J_+ - J_-\right) = \frac{\hbar}{i\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$J_z = \hbar\,\mathrm{diag}(1, 0, -1)$$

4) Rabi's formula

$$\dot{c} = \frac{i}{4}\gamma_1 e^{i\Delta t} d \qquad \Delta = \omega - \omega_0 << \gamma_1$$

$$\dot{d} = \frac{i}{4}\gamma_1 e^{-i\Delta t} c$$

$$\ddot{c} = \left(\frac{i}{4}\gamma_1 (i\Delta) d + \frac{i}{4}\gamma_1 \dot{d}\right) e^{i\Delta t}$$

$$= \frac{i}{4}(i\Delta\gamma_1)\left(\frac{4}{i\gamma_1}\right)\dot{c} + \left(\frac{i}{4}\gamma_1\right)^2 c$$

$$= i\Delta\dot{c} + \left(\frac{\gamma_1}{4}\right)^2 c$$

$$\ddot{c} - i\Delta\dot{c} + \left(\frac{\gamma_1}{4}\right)^2 c = 0$$

let $c = e^{xt}$ $\quad \dot{c} = xe^{xt}$ $\quad \ddot{c} = x^2 e^{xt}$

$$x^2 - i\Delta x + \left(\frac{\gamma_1}{4}\right)^2 = 0$$

$$x_\pm = \frac{1}{2}\left(i\Delta \pm \sqrt{-\Delta^2 - 4\left(\frac{\gamma_1}{4}\right)^2}\right)$$

$$= \frac{i}{2}\left(\Delta \pm \underbrace{\sqrt{\Delta^2 + \frac{\gamma_1^2}{4}}}_{=f}\right) = \frac{i}{2}(\Delta \pm f)$$

$$C(t) = A_+ e^{x_+ t} + A_- e^{x_- t}$$

$$C(t) = e^{i\frac{\Delta t}{2}}\left(A_+ e^{i\frac{ft}{2}} + A_- e^{-i\frac{ft}{2}}\right)$$

at $t=0$, $|C(t)| = \underline{1}$

$$C(t) = e^{i\frac{\Delta t}{2}} \cos\left(\frac{ft}{2}\right)$$

$$d(t) = \int_0^t \frac{i}{4}\gamma_1 e^{-i\frac{\Delta t'}{2}} \cos\left(\frac{ft'}{2}\right) dt'$$

$$\simeq \frac{i}{4}\gamma_1 e^{-i\frac{\Delta t'}{2}} \frac{2}{f} \sin\left(\frac{ft'}{2}\right)\Big|_0^t$$

check $$= i\left(\frac{\gamma_1}{2f}\right) e^{-i\frac{\Delta t}{2}} \sin\left(\frac{ft}{2}\right)$$

$$\dot{C} \simeq \frac{i}{4}\gamma_1 e^{i\Delta t}\left(\frac{i\gamma_1}{2f}\right) e^{-i\frac{\Delta t}{2}} \sin\left(\frac{ft}{2}\right)$$

$$= -e^{+i\frac{\Delta t}{2}} \frac{1}{4}\gamma_1^2 \left(\frac{1}{2f}\right) \sin\left(\frac{ft}{2}\right)$$

$$C(t) = \frac{1}{4}\left(\frac{\gamma_1^2}{f^2}\right) \cos\left(\frac{ft}{2}\right)$$

$$f^2 \simeq \frac{\gamma_1^2}{4}, \quad C(t) = \cos\left(\frac{ft}{2}\right)$$

$$|d|^2 = \frac{\gamma_1^2}{4f^2} \sin^2\left(\frac{ft}{2}\right)$$

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5)

Spin-1 Particle

Spin 1 particle charge q mass m in uniform $\vec{B}$ field.
At $t=0$, particle is in an eigenstate of $\hat{S}_y$.
Find $\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle$ at time t. $\quad H = -\vec{\mu}\cdot\vec{B} = -\left(\frac{qgB}{2mc}\right)\hat{S}_z = -\omega_0 \hat{S}_z$

from $\quad [\hat{S}_y]^z = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $\quad$ solve eigenvalue $[S_y]\begin{pmatrix} \, \end{pmatrix} = \mu\hbar\begin{pmatrix} \, \end{pmatrix}$

see p. 87

$$|\psi(0)\rangle = |1,1\rangle_y \xrightarrow[z]{} \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}$$

$$|\psi(t)\rangle \xrightarrow[z]{} \frac{1}{2}\begin{pmatrix} e^{i\omega_0 t} \\ i\sqrt{2} \\ -e^{-i\omega_0 t} \end{pmatrix}$$

$$\langle S_z \rangle = \langle \psi(t) | \hat{S}_z | \psi(t) \rangle = \begin{pmatrix} \, \end{pmatrix}^* \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \, \end{pmatrix} = 0$$

$$\langle S_y \rangle = \frac{1}{4}\begin{pmatrix} \, \end{pmatrix}^* \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \, \end{pmatrix}$$

$$= \frac{1}{4}\left(\frac{\hbar}{\sqrt{2}}\right) 2(-i\sqrt{2})\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right) = \hbar \sin(\omega_0 t)$$

$$\frac{d}{dt}\langle \hat{S}_y \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{S}_y] \rangle = \frac{i}{\hbar}(-\omega_0)\langle \underbrace{[\hat{S}_z, \hat{S}_y]}_{-i\hbar \hat{S}_x} \rangle$$

$$= -\omega_0 \langle \hat{S}_x \rangle$$

integrating,

$$\langle \hat{S}_y \rangle = \hbar \cos(\omega_0 t)$$

6 Spin-operator transforms as Euclidean vector. $\langle \hat{S}_i \rangle$ are components of spin angular momentum which must transform as Euclidean vector.

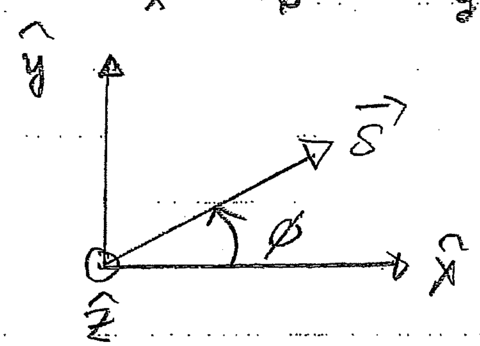
Proof: (see for example Sakurai Quantum Mechanics)

Expand exponentials and use $[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$

$$\hat{R}^{s\dagger}(\phi \hat{z})\, \hat{S}_x\, \hat{R}^s(\phi \hat{z})$$

$$= \exp\left(\frac{i\hat{S}_z\phi}{\hbar}\right) \hat{S}_x \exp\left(\frac{-i\hat{S}_z\phi}{\hbar}\right)$$

$$= \hat{S}_x \cos\phi - \hat{S}_y \sin\phi$$



7) $\langle S_z' \rangle = \frac{\hbar}{2} \cos\theta$

$= \frac{\hbar}{2}(P_+ - P_-)$

$P_\pm$ probability to measure $\pm\frac{\hbar}{2}$ along z' axis

We also have $P_+ + P_- = 1$ so we may write

$P_+ = \cos^2\alpha$ & $P_- = \sin^2\alpha$

Then $\frac{\hbar}{2}(\cos^2\alpha - \sin^2\alpha) = \frac{\hbar}{2}\cos\theta$

From which we find $\alpha = \theta/2$