Phys 521 July 2023

Lec 10: Feynman Path Integral

Propagator

$$\psi(x,t) = \int_{-\infty}^{\infty} \underbrace{\langle x|\hat{U}(t)|x'\rangle}_{=\text{propagator}} \langle x'|\psi(0)\rangle \, dx'$$

here, all integrals assumed to be from $-\infty$ to $+\infty$

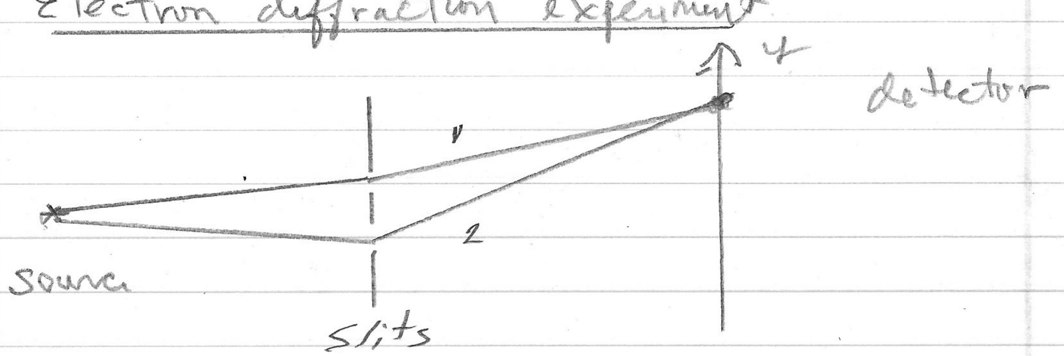
Feynman's conjecture:

$$\langle x_1|\hat{U}(t_1 - t_0)|x_0\rangle \sim \sum_{\text{all paths}} \exp\left(\frac{i}{\hbar} S[x]\right)$$

where $S[x]$ is the classical action evaluated on path (t_0, x_0) to (t_1, x_1).

For macroscopic particle, $\frac{S[\bar{x}]}{\hbar} \gg 1$, $\bar{x}$ being the extremum path, so that the phase varies so rapidly off of $\bar{x}$ and the exponential factors sum to zero except for $x = \bar{x}$. However, quantum particle $S[\bar{x}]/\hbar \sim 1$ so all paths contribute to propagator.

Electron diffraction experiment



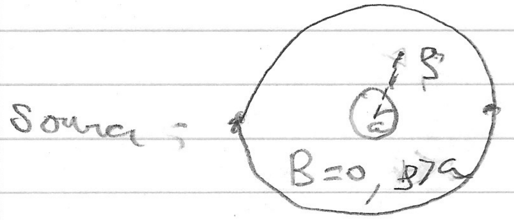
amplitude $\Psi(y) = \Psi_0 \left(e^{iS_1/\hbar} + e^{iS_2/\hbar} \right)$

$$= \Psi_0 e^{iS_1/\hbar} \left(1 + e^{i\Delta S/\hbar} \right)$$

$$P(y) = |\Psi|^2 = \frac{1}{2}\left[1 + \cos\left(\frac{\Delta S}{\hbar}\right)\right]$$

Aharonov - Bohm effect

Consider electrons moving in plane ⊥ long solinoid of radius a along $\hat{z}$ direction



$$\vec{B} = \begin{cases} B_0 \hat{z} & \rho < a \\ 0 & \rho > a \end{cases}$$

$$\mathcal{L}_{EM} = -e\phi + \frac{e}{c}\vec{A}\cdot\vec{v} \qquad q = -e$$

vector potential:

$$\rho > a \qquad \vec{A}_{\rho>a} = \frac{B_0 a^2}{2\rho}\hat{\phi}$$

$$\rho < a \qquad \vec{A}_{\rho<a} = \frac{B_0 \rho}{2}\hat{\phi}$$

$$B_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) = \begin{cases} 0 & \rho > a \\ B_0 & \rho < a \end{cases}$$

$$\psi_{det} = \psi_0 \left(1 + e^{i\Delta S/\hbar}\right)$$

$$\Delta S = \frac{e}{c}\oint \vec{A}\cdot\vec{v}\,dt = \frac{e}{c}\oint \vec{A}\cdot d\vec{s} = \frac{e}{c}\Phi_B$$

1 means 2 gives closed path

magnetic flux $\Phi_B = B_0 \pi a^2$

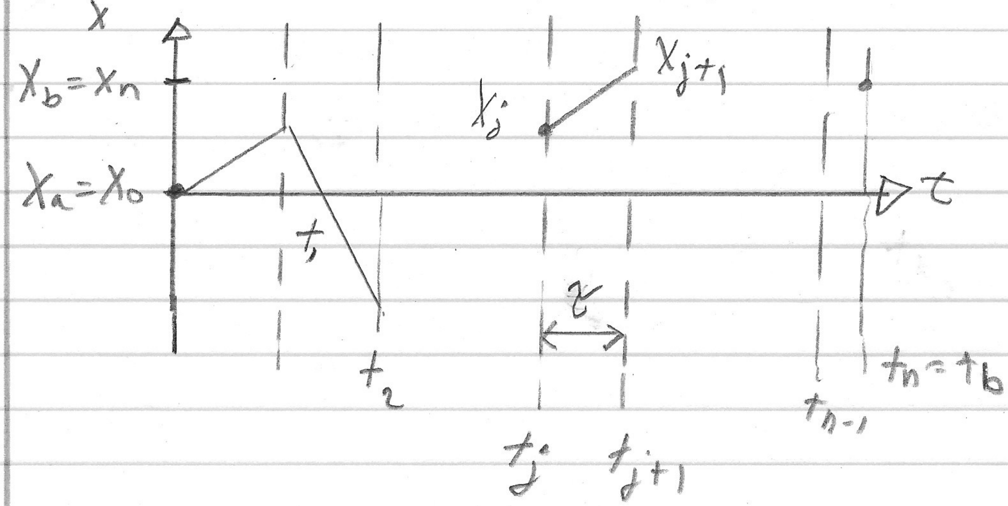
$$P_{det} = \frac{1}{2}\left[1 + \cos\left(\frac{e\Phi_B}{\hbar c}\right)\right]$$

There is no magnetic field (classical force) where electron propagates, but there produces physical effect.

note that Φ_B is <u>gauge invariant</u>.

Feynman Path Integral (FPI)

take limit of discrete spacetime path.



n links

n-1 intermediate times $\quad \tau \equiv \dfrac{t_b - t_a}{n}$

$t_a \equiv t_0$

$t_b = t_n = t_a + n\tau$

propagator $\quad \hat{H} = \dfrac{\hat{p}^2}{2m} + V(\hat{x})$

$$\langle x_b, t_b | x_a, t_a \rangle = \langle x_b | \hat{U}(t_b - t_a) | x_a \rangle$$

$$\hat{U}(t_b - t_a) = \exp\left[-\frac{i}{\hbar}\hat{H}(t_b - t_a)\right]$$

$$= \left(\hat{U}(\tau)\right)^n$$

insert $\quad I = \displaystyle\int_{-\infty}^{\infty} dx_j \, |x_j\rangle\langle x_j|$

n-1 times, for each link

$$\langle x_b, t_b | x_a, t_a \rangle = \int_{-\infty}^{\infty} dx_1 \cdots dx_{n-1} \prod_{j=1}^{n} \langle x_{j+1} | \hat{U}(\tau) | x_j \rangle$$

all integrals go from $-\infty$ to ∞. Paths include those where $\dot{x}$ > speed of light!

Must prove that result is Lorentz Invariant.

for small τ, $\hat{U}(\tau) = 1 - \frac{i\tau}{\hbar} \hat{H}(\hat{x}, \hat{p})$

$$= 1 - \frac{i\tau}{\hbar}\left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right]$$

$V(\hat{x}) | x_j \rangle = V(x_j)$

we need to use $\hat{p}|p\rangle = p|p\rangle$. Insert n identity operators

$$I = \int_{-\infty}^{\infty} dp_j \, |p_j\rangle\langle p_j|$$ as

$$\int dp_j \langle x_{j+1} | \left(1 - \frac{i\tau}{\hbar}\left[\frac{\hat{p}^2}{2m} + V(x_j)\right]\right) | p_j \rangle$$

$$\times \langle p_j | x_j \rangle$$

$$= \int dp_j \left[1 - \frac{i\tau}{\hbar}\left(\frac{p_j^2}{2m} + V(x_j)\right)\right] \langle x_{j+1} | p_j \rangle \langle p_j | x_j \rangle$$

define $E_j = \frac{p_j^2}{2m} + V(x_j)$

approximate $1 - \frac{i\tau}{\hbar} E_j \approx e^{-i\frac{E_j \tau}{\hbar}}$

~~p~~ use explicit plane wave:

$$\langle x_{j+1} | p_j \rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{i}{\hbar} p_j x_{j+1}\right)$$

so

$$\langle x_{j+1} | \hat{U}(\tau) | x_j \rangle = \frac{1}{2\pi\hbar} \int dp_j \, e^{-i E_j \tau/\hbar} e^{\frac{i}{\hbar} p_j x_{j+1}} e^{\frac{-i}{\hbar} p_j x_j}$$

a~~rgum~~ exponential arguments,

$$\{\} = \frac{i\tau}{\hbar}\left[p_j \frac{(x_{j+1} - x_j)}{\tau} - E_j \right]$$

this is the Legendre transformation

$$\dot{x}_j \equiv \frac{x_{j+1} - x_j}{\tau}$$

$$\mathcal{L}(x_j, \dot{x}_j) = p_j \dot{x}_j - E_j = p_j \dot{x}_j - H(x_j, p_j)$$

we will come back to this.

Then with n insertions of $I = \int dp_j |p_j\rangle\langle p_j|$ one for each link,

$$\langle x_b, t_b | x_a, t_a \rangle = \prod_{j=1}^{n-1} \int dx_j \prod_{j=1}^{n} \int \frac{dp_j}{2\pi\hbar}$$

$$\times \prod_{j=1}^{n} \exp\left\{ \frac{i\tau}{\hbar} \left[\frac{p_j (x_{j+1} - x_j)}{\tau} - E_j \right] \right\}$$

now we take limit, $n \to \infty$, $\tau \to 0$

and define "functional measures" $\mathcal{D}x(t)$, $\mathcal{D}p(t)$ limits of $\prod_j dx_j$, $\prod_j \frac{dp_j}{2\pi\hbar}$.

$$\prod_{j=1}^{n} \exp\{\} = \exp \frac{i\tau}{\hbar} \sum_{j=1}^{n} \left[\frac{p_j (x_{j+1} - x_j)}{\tau} - E_j \right]$$

$$\xrightarrow[\lim]{} \exp \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left(p\dot{x} - E(p,x) \right)$$

$$= \exp \frac{i}{\hbar} \int_{t_a}^{t_b} dt \; S(t_b, t_a)$$

giving formal expression, Phase space path integral

$$\langle x_b, t_b | x_a, t_a \rangle = \int_{x_a}^{x_b} \mathcal{D}x(t) \mathcal{D}p(t) \exp\left[\frac{i}{\hbar} S(t_b, t_a) \right]$$

Deriving Feynman's Conjecture

For $E = \frac{p^2}{2m} + V(x)$ the momentum integral can be done explicitly.

$$I_j = \int \frac{dp_j}{2\pi\hbar} \exp\left\{\frac{i\tau}{\hbar}\left[\frac{p_j(x_{j+1}-x_j)}{\tau} - E_j\right]\right\}$$

$$\{\} = \frac{i}{\hbar}\left[p_j(x_{j+1}-x_j) - E_j\tau\right]$$

$$= \frac{i}{\hbar}\left[p_j(x_{j+1}-x_j) + \frac{\tau p_j^2}{2m} - V\tau\right]$$

$$= -\frac{i\tau}{\hbar}V(x_j) - \frac{i\tau}{2m\hbar}\left[p_j^2 + \frac{2m}{\tau}p_j(x_{j+1}-x_j)\right]$$

$$= -\frac{i\tau}{\hbar}V(x_j) - \frac{i\tau}{2m\hbar}\left[p_j - \frac{m}{\tau}(x_{j+1}-x_j)\right]^2$$

$$+\frac{i\tau}{2m\hbar}\left(\frac{m}{\tau}\right)^2(x_{j+1}-x_j)^2$$

in integration, let $p_j' = p_j - \frac{m}{\tau}(x_{j+1}-x_j)$

then with $\int dx \exp(-ax^2) = \sqrt{\frac{\pi}{a}}$

$$a = \frac{i\tau}{2m\hbar}$$

$$I_j = e^{-\frac{i}{\hbar}\tau V(x_j)}\, e^{i\frac{m}{2\hbar}\tau\left(\frac{x_{j+1}-x_j}{\tau}\right)^2} \frac{1}{2\pi\hbar}\sqrt{\frac{m}{2\pi i\hbar\tau}}$$

sum exponential factors in propagator,

$$\langle x_b t_b | x_a t_a \rangle = \prod_{j=1}^{n-1} \left(\int dx_j \sqrt{\frac{m}{2\pi i \hbar \tau}} \right)$$

$$\times \exp\left\{ \frac{i\tau}{\hbar} \sum \left(\frac{m}{2} \left(\frac{x_{j+1} - x_j}{\tau} \right)^2 - V(x_j) \right) \right\}$$

$$\xrightarrow[n\to\infty]{\lim} \int_{x_a}^{x_b} \mathcal{D}' x(t) \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left(\frac{m}{2} \dot{x}^2 - V(x) \right) \right\}$$

$\mathcal{L}(x, \dot{x}) = \frac{m}{2} \dot{x}^2 - V(x)$ and we "close our eyes" and redefine the measure to include $\sqrt{\ }$ factors.

note $\sqrt{\ }$ factors have dimention of $(\text{length})^{-1}$ making $\mathcal{D}' x(t)$ dimentionless.

We now have the path integral

$$\langle x_b t_b | x_a t_a \rangle = \int_{x_a}^{x_b} \mathcal{D}' x(t) \exp\left(\frac{i}{\hbar} S[x] \right)$$

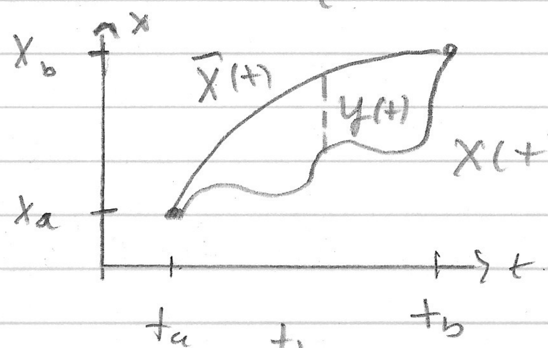
with $S[x] = \int_{t_a}^{t_b} dt \, \mathcal{L}(x, \dot{x})$

Feynman went further for a class of potentials

$$V(x) = a + bx + cx^2 + d\dot{x} + ex\dot{x}$$

where a, b, c, d, e are potentially time dependant.

Expand about classical (extremum) path $\bar{x}(t)$



$$x(t) = \bar{x}(t) + y(t)$$

$$S[x] = S[\bar{x}+y] = \int_{t_a}^{t_b} dt\; \mathcal{L}(\bar{x}+y, \dot{\bar{x}}+\dot{y})$$

$$\mathcal{L}(x,\dot{x}) \simeq \mathcal{L}(\bar{x},\dot{\bar{x}}) + y\frac{\partial \mathcal{L}}{\partial x}\Big|_{\bar{x},\dot{\bar{x}}} + \dot{y}\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big|_{\bar{x},\dot{\bar{x}}}$$

$$+ \frac{1}{2}y^2\frac{\partial^2 \mathcal{L}}{\partial x^2}\Big| + y\dot{y}\frac{\partial^2 \mathcal{L}}{\partial x \partial \dot{x}}\Big| + \frac{1}{2}\dot{y}^2\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2}\Big|$$

terms linear in $y, \dot{y}$ vanish due to extremal condition,

$y = 0$ at t_a, t_b

$$\int dt\left[\dot{y}\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big| + y\frac{\partial \mathcal{L}}{\partial x}\Big|\right] = y\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big|_{t_a}^{t_b}$$

$$+ \int dt\, y\underbrace{\left[\frac{\partial \mathcal{L}}{\partial x}\Big| - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big|\right]}_{=0 \text{ by Euler's equation}} = 0$$

So

$$S[x] = S[\bar{x}] + \int_{t_a}^{t_b} dt \left[\tfrac{1}{2} m \dot{y}^2 - e y \dot{y} - c y^2\right]$$

define

$$A(t_b - t_a) = \int_0^0 \mathcal{D}' y(t) \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt \left(\tfrac{1}{2} m \dot{y}^2 - e y \dot{y} - c y^2\right)\right\}$$

A depends only on the time difference

$$\langle x_b, t_b | x_a, t_a \rangle = A(t_b - t_a)\, e^{\frac{i S[\bar{x}]}{\hbar}}$$

<u>Comments:</u>

① Schrödinger equation derived from path integral (Feynman & Hibbs)

②
$$\langle x_b | \Psi(t_b) \rangle = A(t_b - t_a) \int_{-\infty}^{\infty} dx_a\, e^{\frac{i}{\hbar} S[\bar{x}]} \langle x_a | \Psi(t_a) \rangle$$

factor A comes out of integral.

③ $A(t)$ can be obtained from simple second order linear differential equation.

L.S. Schulman, <u>Techniques and Applications of Path Integrals.</u>

see homework # 7.