Phys 521 Fall 2023

Lec 11: Theory of Angular Momentum

Theory of continuous transformations of vectors, Lie ("lee") groups.

ref. Cahn, *Semi-Simple Lie Algebras and Their Representations*

Group elements

$$g(\vec{\theta}) = \exp(-i\,\theta_i \hat{T}_i) \qquad i = 1, N_g$$

$\hat{T}_i$ are Hermitian operators called generators.

N_g is the dimension (dim) of the group.

Invariance and conservation laws

Group	Generator	invariance ⇒ conservation law
space translations U(1)	p_i	momentum $i = 1, 2, 3$
time translations U(1)	H	energy
rotations SU(2)	$\vec{J}$	angular momentum

U(1) – one dimensional phase $n_g = 1$

SU(2) – unitary, det = 1, 2×2 matrices (lowest dim. representation)

SU(n): $n_g = n^2 - 1$ for SU(2), $n_g = 3$

note translation in different directions commute,

$$\left[\hat{P}_i, \hat{P}_j\right] = \delta_{ij}$$

So each translation is a separate $U(1)$.

$$\hat{T}_x(a) = \exp(-i\, a \hat{P}_x)$$

However, rotations about different axes do not commute.

A <u>representation</u> is a mapping of generators $\hat{T}_i$ to explicit operators acting on an <u>n-dimensional</u> vector space. n is the dimension of the representation.

Almost all group properties are determined by the algebra of the generators.

$$\left[\hat{T}_i, \hat{T}_j\right] = i\, f_{ijk} T_k \quad \text{summation convention!}$$

f_{ijk} = structure constants

lowest dimensional representation is called the fundamental or defining representation

$SU(2)$ - unitary, det = 1 matrices
defining rep. is 2×2 hermitian
Pauli matrices

SU(2) generators $\frac{\sigma_i}{2}$

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\,\varepsilon_{ijk}\,\frac{\sigma_k}{2}$$

SU(2) structure constants

spinor rotation

$$R^s(\vec{\theta}) = \exp\left(-i\,\vec{\theta}\cdot\vec{\sigma}/2\right)$$

$\vec{\sigma}$ transform as components of Euclidean vector.

<u>Irreducible</u> representation of SU(2)
↳ no invariant subgroups

dim	angular momentum eigenvalue	state or particle
$\underset{\sim}{1}$	0	Scalar, Higgs
$\underset{\sim}{2}$	$\frac{1}{2}, -\frac{1}{2}$	spin-½, electron
$\underset{\sim}{3}$	1, 0, -1	spin 1 photon W, Z
$\underset{\sim}{4}$	$-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$	Spin 3/2 gravitino ?
$\underset{\sim}{5}$	-2, -1, 0, 1, 2	Spin 2 graviton?

multiplicity of states mult. $= \underset{\sim}{n}^2 - 1$

Rotation matricies

spinor rotation operator

$$R^s(\theta\hat{y}) = \exp(-i\theta\hat{S}_y/\hbar)$$

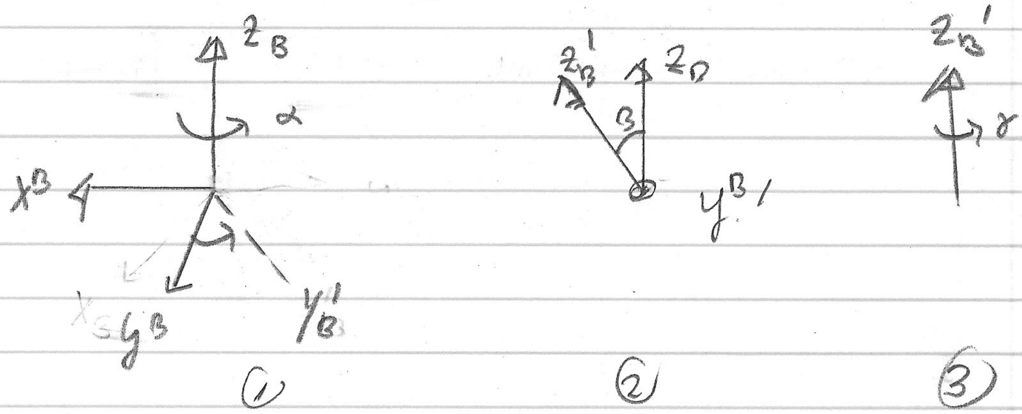
$$\underset{z}{\doteq} \quad I\cos\theta/2 - i\sigma_y \sin\theta/2$$

$$-i\sigma_y = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} = \langle \tfrac{1}{2}, m' | R^s(\theta\hat{y}) | \tfrac{1}{2}, m\rangle$$

$$\equiv \mathcal{D}^{1/2}_{m'm}(R^s(\theta\hat{y}))$$

more generally, in terms of Euler angle of classical mechanics, rotation about rigid body axes



Equivalent to rotations about fixed reference frame axes x, y, z to reverse order,

$$R(\alpha\beta\gamma) = R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)$$

In terms of Euler angles, rotation for general angular momentum state $|j,m\rangle$

$$R^j(\alpha\beta\gamma) = \exp\left(-i\frac{J_z}{\hbar}\alpha\right)\exp\left(-i\frac{J_y}{\hbar}\beta\right)\exp\left(-i\frac{\hat{J}_z}{\hbar}\gamma\right)$$

$$\mathcal{D}^j_{m'm}(\alpha\beta\gamma) = \langle j;m'|\,R^j(\alpha\beta\gamma)\,|j,m\rangle$$

"Wigner function"

On homework you showed:

$$\mathcal{D}^1(\beta) = \frac{1}{2}\begin{bmatrix} 1+\cos\beta & -\sqrt{2}\sin\beta & 1-\cos\beta \\ \sqrt{2}\sin\beta & \cos\beta & -\sqrt{2}\sin\beta \\ 1-\cos\beta & \sqrt{2}\sin\beta & 1+\cos\beta \end{bmatrix}$$

For spinor,

$$\mathcal{D}^{1/2}_{m'm}(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix}\begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}$$

$$= \begin{bmatrix} e^{-i(\alpha+\gamma)/2}\cos\beta/2 & -e^{-i(\alpha-\gamma)/2}\sin\beta/2 \\ e^{+i(\alpha-\gamma)/2}\sin\beta/2 & e^{i(\alpha+\gamma)/2}\cos\beta/2 \end{bmatrix}$$

SU(2) representation theory

denote generic generators as $\frac{J_i}{\hbar}$

start with

$$\left[\frac{J_i}{\hbar}, \frac{J_j}{\hbar}\right] = i\,\varepsilon_{ijk}\,\frac{J_k}{\hbar}$$

Basis states for vector space acted upon by group elements defined by maximal set of commuting operators. For SU(2)

one generator $\frac{\hat{J}_3}{\hbar}$

Casimir operator $\frac{\vec{J}^2}{\hbar^2}$

where $J^2 = J_1^2 + J_2^2 + J_3^2 = \vec{J}\cdot\vec{J}$

commutes with all generators

$$[J^2, J_i] = 0$$

label states by so-far unknown eigenvalues (dimensionless)

$$J^2 \,|\lambda, m\rangle = \hbar^2 \lambda\, |\lambda, m\rangle$$

$$J_3\, |\lambda, m\rangle = \hbar m\, |\lambda, m\rangle$$

First $m^2 \le \lambda$

Since J_i are hermitian, eigenvalues are real and square-eigenvalues ≥ 0

$$\left(\langle \lambda, m|\, J_i\right)\left(J_i\, |\lambda, m\rangle\right) = \langle J_i^2 \rangle \ge 0$$

so
$$\langle \lambda, m|\left(J_1^2 + J_2^2\right)|\lambda, m\rangle \ge 0$$

$$\langle \lambda, m|\left(J^2 - J_3^2\right)|\lambda, m\rangle \ge 0$$

$$\Rightarrow \lambda - m^2 \ge 0$$

Second, introduce raising, lowering operators.

$$J_\pm \equiv J_1 \pm i J_2 \qquad \text{(not Hermitian)}$$

$$J_+^\dagger = J_-$$

$$[J_3, J_\pm] = [J_3, J_1] \pm i\,[J_3, J_2]$$
$$= i\hbar J_2 \pm i(-i\hbar) J_1$$
$$= \pm\hbar\left(J_1 \pm i J_2\right) = \pm\hbar J_\pm$$

$$[J_3, J_\pm] = \pm\hbar J_\pm$$

Eigenvalue of state $J_\pm |\lambda, m\rangle$:

$$J_3\left(J_\pm |\lambda, m\rangle\right) = \left(J_\pm J_3 + \underbrace{[J_3, J_\pm]}_{\pm\hbar J_\pm}\right)|\lambda, m\rangle$$

$$\hbar(m \pm 1)\left(J_\pm |\lambda, m\rangle\right)$$

so
$$J_\pm |\lambda, m\rangle = C_\pm |\lambda, m \pm 1\rangle$$

raises, lowers eigenvalue

$C_\pm$ determined by normalization

$$J_+ J_- = (J_1 + iJ_2)(J_1 - iJ_2) = J^2 - J_3^2 + \hbar J_3$$

$$J_- |\lambda, m\rangle = C_- |\lambda, m\rangle$$

$$\langle \lambda, m | J_+ J_- |\lambda, m\rangle = |C_-|^2$$

$$|C_-|^2 = \langle \lambda, m | \left(J^2 - J_3^2 + \hbar J_3\right) |\lambda, m\rangle$$

$$= \hbar^2 (\lambda - m^2 + m)$$

$$C_- = \hbar\sqrt{\lambda - m(m-1)}$$

similarly
$$C_+ = \hbar\sqrt{\lambda - m(m+1)}$$

There must be a max and min value for m.
(finite dimension of rep)

define $m_{max} = j$ $\quad m_{min} = -j$

then
$$J_+ |\lambda, j\rangle = 0$$

$$J_- J_+ |\lambda, j\rangle = (J^2 - J_3^2 - \hbar J_3) |\lambda, j\rangle$$

compare to $J_+ J_-$

$$= \hbar^2 (\lambda - j^2 - j) |\lambda, j\rangle = 0$$

$$\lambda = j(j+1)$$

customary to label state as $|j, m\rangle$.

$$J_\pm |j, m\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)}\ |j, m \pm 1\rangle$$

dim of rep $n = 2j + 1$

"Addition" of angular Momentum

System with two angular momentum operators in different subspaces. For example H-atom

$$|\psi\rangle = |n, \ell, m\rangle |\tfrac{1}{2}, m_s\rangle$$

adding orbital $\vec{L}$ and spin $\vec{S}$

$$\vec{J} = \vec{L} + \vec{S}$$

Eigenstates of total J^2, J_z?
There will be linear combinations of states

$$|\ell, m\rangle |\tfrac{1}{2}, m_s\rangle$$

These linear combinations $|j, m\rangle$ will not mix with other combinations. They form irreducible representations of SU(2).
(irreps)

General problem:

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

Irreps of SU(2) follow simple rule:

$$\boxed{|j_1 - j_2| \le j \le j_1 + j_2}$$

states $|j_1, m_1\rangle$ $\quad -j_1 \le m_1 \le +j_1$

multiplicity (mult.) $2j_1+1$

multiplicity of direct product

$$|j_1, m_1\rangle |j_2, m_2\rangle \qquad (2j_1+1)(2j_2+1)$$

Taking $j_2 > j_1$ we have the rearrangement,

$$\sum_{j=j_2-j_1}^{j_2+j_1} (2j+1) = (2j_1+1)(2j_2+1)$$

number of states does not change in irreducible decomposition.

Two ways of writing decomposition: j-value or multiplicity.

Example $\qquad j_1 = 1 \qquad j_2 = 2 \quad (2+1)(4+1) = 15$ states

$$|2-1| \le j \le 2+1 \implies j = 1, 2, 3$$

in terms of j:

$$\underset{\sim}{1} \times \underset{\sim}{2} = \underset{\sim}{1} + \underset{\sim}{2} + \underset{\sim}{3}$$

in terms of mult.

$$\underset{\sim}{3} \times \underset{\sim}{5} = \underset{\sim}{3} + \underset{\sim}{5} + \underset{\sim}{7}$$

Explicit linear combination given by orthogonal, real (convention) Clebsch-Gordon matricies

$$|jm, j_1 j_2\rangle = \sum_{m_1, m_2} |j_1 m_1, j_2 m_2\rangle \underbrace{\langle j_1 m_1, j_2 m_2 | jm\rangle}_{\text{CG coefficients}}$$

They can be looked up in table or directly found from raising, lowering operators.

$$J_\pm = J_1 \pm i J_2$$

Example $j_1 = 1/2 \quad j_2 = 1/2$

by j: $\underset{\sim}{\tfrac{1}{2}} \times \underset{\sim}{\tfrac{1}{2}} = \underset{\sim}{0} + \underset{\sim}{1}$ or $\underset{\sim}{2} \times \underset{\sim}{2} = \underset{\sim}{1} + \underset{\sim}{3}$ dimension

$j = 1$ (3 dim) triplet Highest m-value (weight) is unique

$$|1,1\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle_2 \equiv |\underset{1}{\tfrac{1}{2}} ; \underset{2}{\tfrac{1}{2}}\rangle \text{ by order}$$

$$J_- |1,1\rangle = \hbar \sqrt{1(1+1) - (1-1)}\, |1,0\rangle = \sqrt{2}\, |1,0\rangle$$

$$J_{1-} |\tfrac{1}{2}, \tfrac{1}{2}\rangle_1 = \hbar \sqrt{\underset{3/4}{\tfrac{1}{2}(\tfrac{1}{2}+1)} \underset{+1/4}{- \tfrac{1}{2}(\tfrac{1}{2}-1)}}\, |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

where we recall,

$$J_\pm |j, m\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)}\, |j, m \pm 1\rangle$$

note $\quad J_+ |j, j\rangle = 0$

$$J_- |j, -j\rangle = 0$$

then

$$J_-|1,1\rangle = (J_{1-} + J_{2-})|\tfrac{1}{2};\tfrac{1}{2}\rangle$$

$$\sqrt{2}\,|1,0\rangle = |\tfrac{1}{2};-\tfrac{1}{2}\rangle + |-\tfrac{1}{2};\tfrac{1}{2}\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2};-\tfrac{1}{2}\rangle + |-\tfrac{1}{2};\tfrac{1}{2}\rangle\right)$$

$$J_-|1,0\rangle = \hbar\sqrt{1(1+1)-0}\,|1,-1\rangle = \sqrt{2}\,|1,-1\rangle$$

$$\sqrt{2}\,|1,-1\rangle = \frac{1}{\sqrt{2}}\left(2|-\tfrac{1}{2};-\tfrac{1}{2}\rangle\right)$$

$$|1,-1\rangle = |-\tfrac{1}{2};-\tfrac{1}{2}\rangle$$

We get properly normalized triplet states. The rotationally invariant singlet is the orthogonal,

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|+\tfrac{1}{2};-\tfrac{1}{2}\rangle - |-\tfrac{1}{2};+\tfrac{1}{2}\rangle\right)$$

<u>antisymmetric</u>

note that elements of irrep. have same symmetry under <u>exchange of particle label</u>!

$|1,m\rangle$ <u>states are symmetric</u>

Proton magnetic resonance

$$H = -\vec{\mu}\cdot\vec{B} \qquad \vec{\mu} = \left(\frac{e g_p}{2 m_p c}\right)\frac{\hbar}{2}\vec{\sigma}$$

define $\vec{\gamma} = \dfrac{e g_p}{2 m_p c}\vec{B}$ $\qquad g_p = 5.59$

$$H = -\frac{\hbar}{2}\vec{\gamma}\cdot\vec{\sigma}$$

$$\vec{B} = B_1 \cos\omega t\,\hat{x} + B_0\,\hat{z}$$

$$H = -\frac{\hbar}{2}\gamma_1 \cos\omega t\,\sigma_x - \frac{\hbar}{2}\gamma_0\,\sigma_z$$

$$= -\frac{\hbar}{2}\begin{pmatrix} \gamma_0 , & \gamma_1 \cos\omega t \\ \gamma_1 \cos\omega t , & -\gamma_0 \end{pmatrix}$$

remove simpl time dependence

In spinor basis,

$$|\psi(t)\rangle \doteq \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} c(t)\, e^{i\gamma_0 t/2} \\ d(t)\, e^{-i\gamma_0 t/2} \end{pmatrix}$$

cancel

$$i\hbar\left(\dot{c} + \frac{i\gamma_0}{2}c\right)e^{i\gamma_0 t/2} = -\frac{\hbar}{2}\left(\gamma_0 c e^{i\gamma_0 t/2} + \gamma_1 \cos\omega t\, d e^{-i\gamma_0 t/2}\right)$$

$$i\hbar\,\dot{c}\, e^{i\gamma_0 t/2} = -\frac{\hbar}{2}\gamma_1 \cos(\omega t)\, d e^{-i\gamma_0 t/2}$$

$$\boxed{i\dot{c} = -\frac{\gamma_1}{2}\cos(\omega t)\, d e^{-i\gamma_0 t}}$$

and

$$\boxed{i\dot{d} = -\frac{\gamma_1}{2}\cos(\omega t)\, c e^{+i\gamma_0 t}}$$

expand $\cos\omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$

define $\Omega \equiv \omega + \gamma_0 \qquad \Delta \equiv \omega - \gamma_0$

$$\dot{c} = \frac{i\gamma_1}{4}\left(e^{i\Delta t} + e^{-i\Omega t}\right) d$$

$$\dot{d} = \frac{i\gamma_1}{4}\left(e^{i\Omega t} + e^{-i\Delta t}\right) c$$

near resonance $\omega \sim \gamma_0$, $e^{i\Omega t}$ oscillates rapidly compared to $e^{i\Delta t}$ and can be neglected.

$$\boxed{\begin{aligned} \dot{c} &\cong \frac{i\gamma_1}{4} e^{i\Delta t} d \\ \dot{d} &\cong \frac{i\gamma_1}{4} e^{-i\Delta t} c \end{aligned}}$$

At resonance $(\Delta = 0)$ we can easily solve

$$\ddot{c} = \frac{i\gamma_1}{4}\dot{d} = -\left(\frac{\gamma_1}{4}\right)^2 c$$

$$c(t) = c(0)\cos\frac{\gamma_1 t}{4} + d(0)\sin\frac{\gamma_1 t}{4}$$
$$d(t) = d(0)\cos\frac{\gamma_1 t}{4} + c(0)\sin\frac{\gamma_1 t}{4}$$

Prob to make transition from up to down $(\Delta = 0)$

$$P(t) = \sin^2\left(\frac{\gamma_1 t}{4}\right)$$

Off resonance, get Rabi's formula for the transition probability $T(t)$

$$T(t) = |d|^2 = \frac{\gamma_1^2}{4f^2} \sin^2\left(\frac{ft}{2}\right)$$

$$f^2 = (\gamma_0 - \omega)^2 + \frac{\gamma_1^2}{4}$$

Amplitude is a Breit Wigner resonance curve:

$\frac{\gamma_1^2}{4f}$