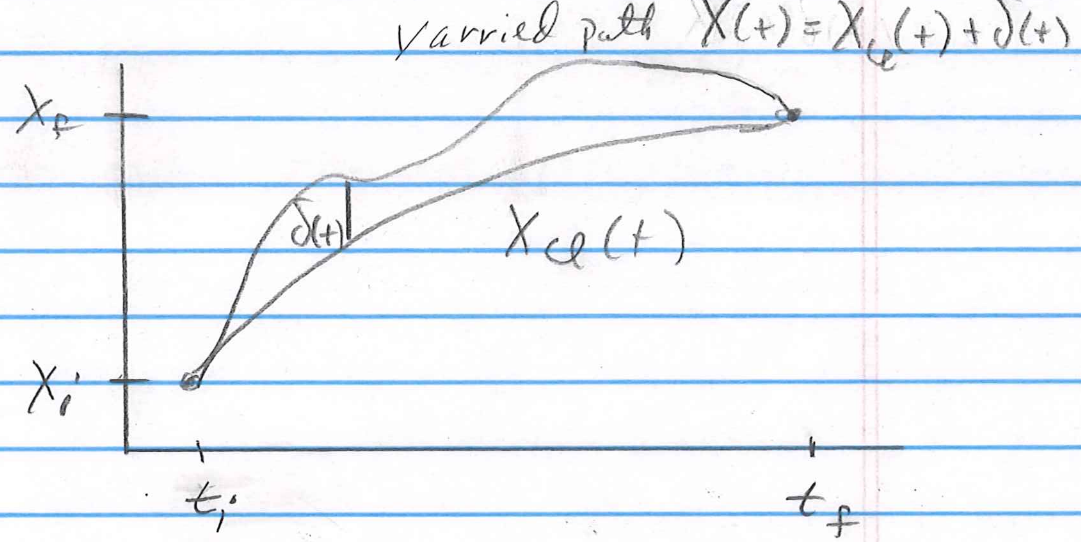
Lec II. Some Classical

Lagrangian

$$\mathcal{L}(x, \dot{x}) = T - V$$ omit time dependent V.

Start with $T = \frac{1}{2} m \dot{x}^2$



$\delta(t_i) = \delta(t_f) = 0$ end points fixed

Variation $\delta(t)$ compares paths at the same time.

Classical path is extremum of action integral,

$$S[x(t)] = \int_{t_i}^{t_f} \mathcal{L}(x, \dot{x})\, dt.$$

S is <u>functional</u> of $x(t)$, $\mathcal{L}$ function $x(t)$, $\dot{x}(t)$

$$\delta S = S[x_{cl}+\delta] - S[x_{cl}]$$

$$= \frac{\delta S}{\partial x}\Big|_{x_{cl}} \delta + \frac{\partial S}{\partial \dot{x}}\Big|_{x_{cl}} \frac{d\delta}{dt}$$

$$\delta S = \int_{t_i}^{t_f} \left[\frac{\partial \mathcal{L}}{\partial x} \delta + \frac{\partial \mathcal{L}}{\partial \dot{x}} \frac{d\delta}{dt} \right] dx$$

integrate by parts with $\delta(t_i) = \delta(t_f) = 0$

$$\delta S = \int_{t_i}^{t_f} \left[\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \right] \delta(t)\, dt$$

Since $\delta(t)$ is arbitrary, we get Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

this gives classical equations of motion. Classically, not interested in $S[x_{cl}]$.

ball falling from height d

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - mgx \qquad x(t) = -\frac{1}{2}gt^2 + d$$

$$(x_0, t_0) = (d, 0)$$
$$(x_1, t_1) = \left(0, \sqrt{\frac{2d}{g}}\right)$$

$$S[\bar{x}] = \int_0^{\sqrt{\frac{2d}{g}}} \left(\frac{m\dot{x}^2}{2} - mgx\right) dt$$

$$= \int_0^{\sqrt{2d/g} = t_1} \left[\frac{m}{2}(-gt)^2 - mg\left(d - \frac{gt^2}{2}\right)\right] dt$$

$$= \int_0^{t_1} (mg t^2 - mgd)\, dt = \frac{1}{3}mg t_1^3 - mg t_1$$

$$= \sqrt{\frac{2g}{d}}\left[\frac{1}{3}mg^2\left(\frac{2d}{g}\right) - mgd\right] = -\frac{1}{3}md\sqrt{2dg}$$

dim $kg\, m^2/s = J \cdot s$

take $m = 1g$, $d = 10\,cm$, $g = 10\,m/s^2$

$$(10^{-3}kg)(0.1m)\sqrt{2(0.1m)\,10\,m/s^2} = \sqrt{2}\,10^{-4}\ J \cdot s$$

Is this small or large?

$$\frac{S[\bar{x}]}{\hbar} = \frac{\sqrt{2}\,10^{-4}\ J \cdot S}{6.6 \times 10^{-34}\ J \cdot s} \sim 10^{30}$$

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However for accelerated electron $\boxed{a = \frac{e\mathcal{E}}{m_e}}$

$$S[\bar{x}] = -\frac{1}{3} m d \sqrt{2d\left(\frac{e\mathcal{E}}{m}\right)}$$

$$KE\,(eV) = e\,\mathcal{E}\,d$$

volts/meter

$$S[\bar{x}] = -\frac{1}{3} d \sqrt{2mKE}$$

$$\frac{\bar{S}}{\hbar} = -\frac{1}{3}\frac{d}{\hbar c}\sqrt{2mc^2 KE} = -\frac{1}{3}\frac{d[nm]}{200}10^3\sqrt{KE}$$

$$\hbar c = 200\ eV \cdot nm$$

$$2mc^2 = 10^6\ eV$$

$$= -\frac{10}{6} d[nm]\sqrt{KE}$$

Q.M. object can have $\frac{S}{\hbar} \sim 1$

Legendre Transformation

See Gelfand and Fomin p. 72. Also, from: "Making Sense of the Legendre Transform", Zia, Redish, McKay, *arXiv*:0806.1147v2, 2009 (figures are taken from here).

In classical mechanics the Lagrangian $\mathcal{L}(\dot{q}, q, t)$ and the Canonical momentum is $p = \partial\mathcal{L}/\partial\dot{q}$. Thus p is the slope of $\mathcal{L}$. For simplicity, consider the case where the potential $V(q)$ and not $\dot{q}$, then $\dot{q}$ appears only in the kinetic energy which is a quadratic in $\dot{q}$. Thus, $\mathcal{L}$ is a convex function of $\dot{q}$.

Putting this into a generic, mathematical language, $\dot{q} \to x$ and $\mathcal{L}(\dot{q}) \to F(x)$ and write the slope of F(x) as $s(x) = dF/dx$. F being convex implies that $s(x)$ can be inverted to get $x(s)$ (Fig. 1)

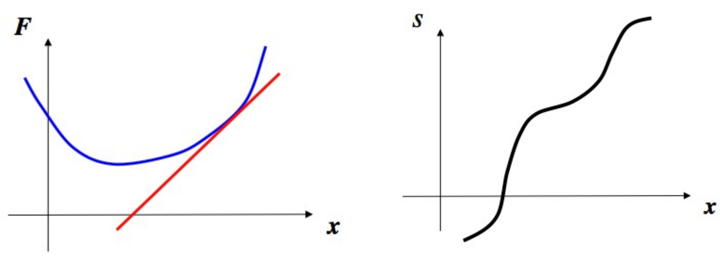

Figure 1: Convex function F

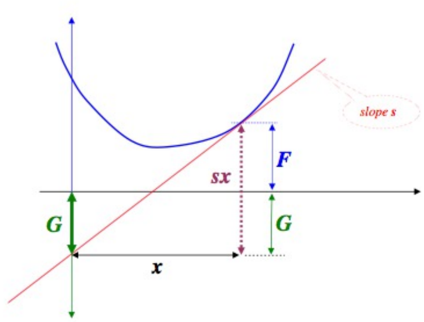

Figure 2: Graphical representation of the relation $(F + G)/x = s$ or $xs = F + G$

The function $G(s)$ contains the same information as $F(x)$ (Fig.2).

$$dG = -\frac{dF}{dx}dx + sdx + xds = xds$$

implies $x(s) = \frac{dG}{ds}$. Or, back to physics language, G is the Hamiltonian $H(p)$ and $\dot{q}(p) = \partial H/\partial p$.

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Hamiltonian

Legendre transformation coordinates q_i
momentum p_i

$$H(q_i, p_i) + \mathcal{L}(q_i, \dot{q}_i) = \sum_i \dot{q}_i p_i$$

where canonical momentum $\boxed{p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}}$

Hamilton's equations -

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \; ; \; -\frac{\partial H}{\partial q_i} = -\dot{p}_i$$

In simple case, $\mathcal{L} = \frac{1}{2} m \dot{x}^2 - V$ $\quad p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$

$$H = \dot{x}(m\dot{x}) - \left(\frac{1}{2} m \dot{x}^2 - V\right) = \frac{p^2}{2m} + V(x)$$

"energy"

generally, Legendre transformation is non-trivial In particular, for E&M

Charge q in classical electric $\vec{\mathcal{E}}$ and magnetic $\vec{B}$ field

$$\vec{F} = q\left(\vec{\mathcal{E}} + \frac{1}{c}\vec{v} \times \vec{B}\right)$$

Lorentz force
c.g.s units

$$\vec{\mathcal{E}} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Nown rewrite F in terms of scalar and vector potentials

$\vec{v} \times \vec{B}$ piece of Lorentz Force

$$\vec{v}\times\vec{B} = \vec{v}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\vec{\nabla})\vec{A}$$

$$F_i = -q\frac{\partial}{\partial x_i}\phi - \frac{q}{c}\frac{\partial \vec{A}}{\partial t} + \frac{q}{c}\frac{\partial}{\partial x_i}(\vec{v}\cdot\vec{A}) - \frac{q}{c}(\vec{v}\cdot\vec{\nabla})A_i$$

$$\frac{dA_i}{dt} = (\vec{v}\cdot\vec{\nabla})A_i + \frac{\partial A_i}{\partial t}$$ total derivative

$$F_i = -q\frac{\partial \phi}{\partial x_i} - \frac{q}{c}\frac{dA_i}{dt} + \frac{q}{c}\vec{v}\cdot\frac{\partial \vec{A}}{\partial x_i}$$

derivable from time dependent potential,

$$V = q\phi - \frac{q}{c}(\vec{v}\cdot\vec{A})$$

$$F_i = -\frac{dV}{dx_i} = -\frac{\partial V}{\partial x_i} - \frac{d}{dt}\frac{\partial V}{\partial \dot{x}_i}$$

$$\frac{\partial V}{\partial \dot{x}_i} = -\frac{q}{c}\frac{\partial}{\partial \dot{x}_i}\left(\sum_j \dot{x}_j A_j\right) = -\frac{q}{c}A_i$$

$$\frac{\partial V}{\partial x_i} = q\frac{\partial \phi}{\partial x_i} + \frac{q}{c}\vec{v}\cdot\frac{\partial \vec{A}}{\partial x_i}$$

V checks.

then Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2} m \dot{x}^2 - q\phi + \frac{q}{c} \vec{v} \cdot \vec{A}$$

canonical momentum

$$P_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m\dot{x}_i + \frac{q}{c} A_i$$

Hamiltonian is then the Legendre transformation

$$H = \sum_i \dot{x}_i \left(m\dot{x}_i + \frac{q}{c} A_i \right) - \frac{1}{2} m \dot{x}_i^2$$

$$+ q\phi - \frac{q}{c} \vec{v} \cdot \vec{A}$$

$$= \sum \frac{1}{2} m \dot{x}^2 + q\phi$$

but $\dot{x}_i = \frac{1}{m} \left(P_i - \frac{q}{c} A_i \right)$

$$\boxed{H = \sum_i \frac{1}{2m} \left(P_i - \frac{q}{c} A_i \right)^2 + q\phi}$$

Useful relation for action : lec 2-8

$$S_{cl}(x_f, t_f; x_i, t_i) = \int_{t_i}^{t_f} \mathcal{L}\, dt$$

$$\frac{\partial}{\partial t_f} S_{cl} = -H(t_f) = -E$$ assuming H conserved

change in S_{cl} due to change in endpoint t_f by Δt

$$\delta S_{cl} = \delta \int_{t_i}^{t_f} \mathcal{L}\, dt + \mathcal{L}(t_f)\Delta t$$

$$= \int_{t_i}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial x} \delta + \frac{\partial \mathcal{L}}{\partial \dot{x}} \frac{d\delta}{dt} \right) dt + \mathcal{L}(t_f)\Delta t$$

as before, integrate by parts but $\delta(t_f) = -\dot{x}(t_f)\Delta t$

proof on next page

$$\delta S_{cl} = -\int \left(\frac{\partial \mathcal{L}}{\partial x} + \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta\, dt$$

end-point term

$$+ \delta(t_f)\frac{\partial \mathcal{L}}{\partial \dot{x}} + \mathcal{L}(t_f)\Delta t$$

First term $= 0$ by Euler Lagrange

$$\delta S_{cl} = -\dot{x}(t_f)\Delta t \left.\frac{\partial \mathcal{L}}{\partial \dot{x}}\right|_{t_f} + \mathcal{L}(t_f)\Delta t$$

see next page

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = p \quad \text{so} \quad \dot{x}p - \mathcal{L} = H$$

$$\delta S_{cl} = -H(t_f)\Delta t \; ; \quad \boxed{\frac{\partial S_{cl}}{\partial t_f} = -H(t_f)}$$

Note on $\delta(t_2)$

$$X(t) = \bar{X}(t) + \delta(t) \qquad \bar{X} = \text{classical path}$$

compare $X(t_2 + \Delta t)$ to $X(t_2)$

keeping $\delta(t_2 + \Delta t) = 0$ and $\bar{X}(t_2 + \Delta t) = \bar{X}(t_2)$

$$X(t_2) = \bar{X}(t_2) + \delta(t_2)$$

$$X(t_2 + \Delta t) = \bar{X}(t_2 + \Delta t) + \overset{0}{\overset{\shortparallel}{\delta(t_2 + \Delta t)}}$$

$$X(t_2 + \Delta t) - X(t_2) = -\delta(t_2)$$

$$\underline{\dot{X}(t_2)\Delta t = -\delta(t_2)}$$

Variation of endpoint: