Phys 521 Fall 2023

Lec 3: Some Probability

1. Probability Axioms (Kolmogorov)

set S "sample space"
subsets $A, B, \ldots$
every subset A $P(A) \geq 0$

disjoint $A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$

$P(S) = 1$

Conditional P of A given B $P(A|B)$

$$P(A|B) \equiv \frac{P(A \cap B)}{P(B)}$$

Bayes Theorem

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

law of total probability B disjoint with A_i
and $\cup A_i = S$

$$P(B) = \sum_i P(B/A_i)\,P(A_i)$$

$$P(A) \simeq \frac{\#\text{ outcomes } A}{\text{total trials}}$$ experimental definition

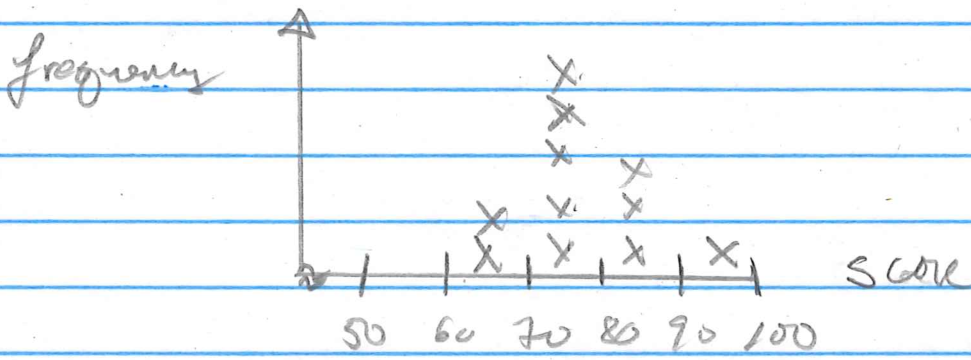
$$P(A+B) \leq P(A) + P(B)$$

equality if A, B disjoint

$$0 < P < 1$$

2. Histogram frequency distribution

Consider exam grades $\{g_i\}$ N students
grade distribution visualized by binning



bin	center	number
7	60.5	2
8	70.5	5
9	80.5	3
10	90.5	1

mean, expectation value

$$\langle x \rangle = \frac{1}{N} \sum N_i x_i$$

Variance

$$(\Delta x)^2 = \frac{1}{N} \sum N_i (x_i - \langle x \rangle)^2$$

Assume some underlying probability

$$p_i \approx \frac{N_i}{N} \qquad \sum p_i = 1$$

$$\langle x \rangle = \sum p_i x_i$$
$$\langle x^2 \rangle = \sum p_i x_i^2$$

Variance $\langle (x - \langle x \rangle)^2 \rangle = \sum p_i \left(x_i - \langle x \rangle \right)^2$

$$= \sum p_i \left(x_i^2 - 2x_i \langle x \rangle + \langle x \rangle^2 \right)$$

$$= \sum_i p_i x_i^2 - 2\langle x \rangle \sum_i p_i x_i + \langle x \rangle^2 \sum p_i$$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

3. Continuous Limit

probabilty density function

$$p_i \longrightarrow p(x)\,dx \qquad \text{PDF}$$

$$\int p(x)\,dx = 1$$

$$\langle x \rangle = \int p(x)\,x\,dx$$

Example: exponential decay

$$\frac{dN}{dt} = -\frac{N(t)}{\tau} \Rightarrow N(t) = N(0)\,e^{-t/\tau}$$

normalized PDF $\qquad p(t) = 0 \quad t < 0$

$$p(t) = \frac{1}{\tau} e^{-t/\tau} \quad t > 0$$

$$\langle t \rangle = \frac{1}{\tau}\int_0^\infty e^{-t/\tau}\, t\, dt = \tau \int_0^\infty e^{-x}\, x\, dx = \tau$$

useful integral : $\int_0^\infty x^n e^{-x} dx = n!$

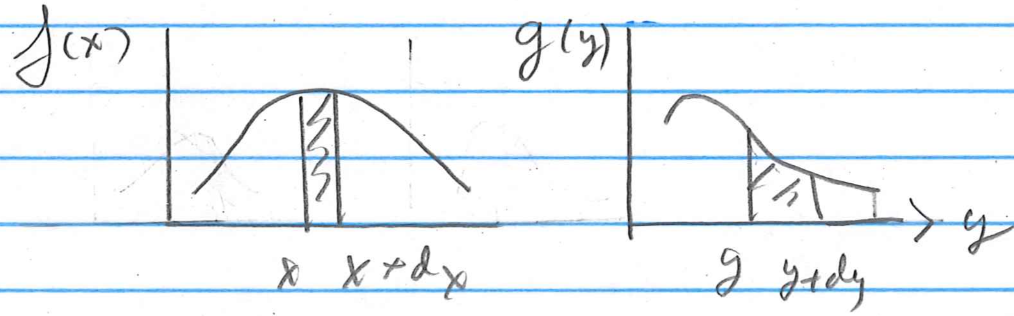
Variance $(\Delta t)^2 = \frac{1}{\tau}\int_0^\infty e^{-t/\tau}(t-\tau)^2 dt$

$$= \tau^2 \int_0^\infty e^{-x}(x-1)^2 dx = \tau^2[2! - 2 + 1] = \tau^2$$

$\Delta t = \tau$

4 **Change of variables** $y = y(x)$

invert $X(y) = x$



$$f(x)\,dx = g(y)\,dy$$

$$g(y)\,dy = \left|\frac{dy}{dx}\right|^{-1} dy\, f(X(y))$$

PDF in y is $\left|\frac{dy}{dx}\right|^{-1} f(X(y))$

<u>example</u> $f(x) = e^{-x}$

$$y = e^{-x}$$
$$x(y) = -\ln y$$

$$\left|\frac{dy}{dx}\right| = e^{-x} = y$$

$$g(y) = \left|\frac{dy}{dx}\right|^{-1} f(x(y))$$

$$= \frac{1}{y} e^{\ln y} = 1 \quad \text{flat}$$

range $x\,[0,\infty]$ $\quad y\,[0,1]$

5 <u>Correlation</u>: two random variables x_1, x_2 and quantity $q(x_1, x_2)$

$$\int P(x_1, x_2)\, dx_1\, dx_2 = 1$$

$$\langle q \rangle = \int P(x_1, x_2)\, q(x_1, x_2)\, dx_1\, dx_2$$

expand q about $\langle x_1 \rangle, \langle x_2 \rangle$

$$q(x_1, x_2) = q(\langle x_1 \rangle, \langle x_2 \rangle) + \left.\frac{\partial q}{\partial x_1}\right|(x_1 - \langle x_1 \rangle)$$

$$+ \left.\frac{\partial q}{\partial x_2}\right|(x_2 - \langle x_2 \rangle)$$

then $\langle q \rangle = q(\langle x_1 \rangle, \langle x_2 \rangle)$

$$(\Delta q)^2 = \langle (q - \langle q \rangle)^2 \rangle$$

$$= \left(\frac{\partial q}{\partial x_1}\Big|\right)^2 \langle (x_1 - \langle x_1 \rangle)^2 \rangle + \left(\frac{\partial q}{\partial x_2}\Big|\right)^2 \langle (x_2 - \langle x_2 \rangle)^2 \rangle$$

$$= 2 \frac{\partial q}{\partial x_1}\Big| \frac{\partial q}{\partial x_2}\Big| \underbrace{\langle (x_1 - \langle x_1 \rangle)(x_2 - \langle x_2 \rangle) \rangle}_{Cov(x_1, x_2) \text{ covariance}}$$

giving

$$(\Delta q)^2 = \left(\frac{\partial q}{\partial x_1}\Big|\right)^2 \Delta x_1^2 + \left(\frac{\partial q}{\partial x_2}\Big|\right)^2 \Delta x_2^2$$

$$+ 2 \frac{\partial q}{\partial x_1}\Big| \frac{\partial q}{\partial x_2}\Big| Cov(x_1, x_2)$$

if x_1, x_2 are uncorrelated

$$P(x_1, x_2) = P_1(x_1) P_2(x_2)$$

and $Cov(x_1, x_2) = 0$

linear correlation coefficient $-1 \leq r \leq 1$

$$r = \frac{\langle x_1 x_2 \rangle - \langle x_1 \rangle \langle x_2 \rangle}{\Delta x_1 \, \Delta x_2} = \frac{Cov(x_1, x_2)}{\Delta x_1 \, \Delta x_2}$$