Lecture 4 - postulates

① Dynamical system represented by ray in Hilbert space.

Overall phase not physical. proper state normalized

$$\langle \psi | \psi \rangle = 1$$

We will make use of un-normalizable plane wave states

$$\langle x | p \rangle = \phi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right)$$

Quantum state $|\psi\rangle$ includes everything it is possible to observe about the system and no more.

normalized as
$$\langle p' | p \rangle = \delta(p' - p)$$

② Unitary time evolution operator

$$|\psi(t)\rangle = \hat{U}(t, t_0) |\psi(t_0)\rangle$$

$$\hat{U}^{+}(t, t_0)\hat{U}(t, t_0) = \hat{I} \qquad \langle \psi(t)|\psi(t)\rangle = \langle \psi(t_0)|\psi(t_0)\rangle$$

For infinitessimal time

$$\hat{U}(t + dt, t) = I - \frac{i}{\hbar}\hat{A}\,dt \xrightarrow[dt \to 0]{} I$$

$\hat{H}$ has dimensions of energy and is the Hamiltonian operator. In classical mechanics H generates translations in time.

$\hat{U}(t)$ evolution equation. We must be careful because we might have $H(t)$ and perhaps even

$$[\hat{H}(t_1), \hat{H}(t_2)] \neq 0$$

Composition property $\quad t_2 > t_1 > t_0$

$$\hat{U}(t_2, t_0) = \hat{U}(t_2, t_1)\,\hat{U}(t_1, t_0)$$

$$\hat{U}(t+\Delta t, t_0) = \hat{U}(t+\Delta t, t)\,\hat{U}(t, t_0)$$

$$= \left(\underline{I} - \frac{i}{\hbar}\hat{H}\Delta t\right)\hat{U}(t, t_0)$$

$$\frac{\hat{U}(t+\Delta t, t_0) - \hat{U}(t, t_0)}{\Delta t} = -\frac{i}{\hbar}\hat{H}\,\hat{U}(t, t_0)$$

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t, t_0) = \hat{H}\,\hat{U}(t, t_0)$$

true very generally. Therefore state evolves as

$$|\psi(t)\rangle = \hat{U}(t, t_0)\,|\psi(t_0)\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle$$

Suppose H has no explicit time dependence.

Then

$$\hat{U}(t) = \lim_{n\to\infty}\left[1 - \frac{i}{\hbar}\hat{H}\frac{t}{N}\right]^N = e^{-i\hat{H}t/\hbar}$$

Second, suppose $[H(t_1), H(t_2)] = 0$. then

$$\hat{U}(t) = \exp\left(\frac{-i}{\hbar}\int_0^t \hat{H}(t')\,dt'\right)$$

Comment on sign of approximation:

$$\hat{O}(t+dt, t) \cong 1 - \frac{i}{\hbar}\hat{H}\,dt$$

this minus sign

gives

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

then for free particle you show:

$$\frac{d}{dt}\langle x\rangle = \frac{1}{m}\langle p\rangle$$

so time evolution has correct sign.

if not, need to define time ordered product
(Shankar, p. 148)

For $\hat{H}$ independent of time get simple

$$\hat{U}(t, t_0) = \exp \frac{-i}{\hbar} \hat{H}(t - t_0)$$ $\hat{H}$ independent of time

③ Observables correspond to Hermitian operators $\hat{\Theta}$ with real eigenvalues θ_i.
Form complete set

$$\hat{\Theta} |\Theta_i\rangle = \Theta_i |\Theta_i\rangle \quad \Theta_i \text{ real eigenvalue}$$

$$|\psi\rangle = \sum_i |\hat{\theta}_i\rangle\langle\hat{\theta}_i|\psi\rangle$$

measurement of $\hat{\Theta}$ yields eigenvalue θ_i with probability

$$P_i = |\langle\theta_i|\psi\rangle|^2$$

"collapse Postulate"

Following measurement, state collapses to eigenstate

Born rule

$$|\psi\rangle \longrightarrow |\theta_i\rangle$$

subsequent measurements of $\hat{\Theta}$ give θ_i with probability 1.

repeated measurements of $\hat{O}$ on identically prepared states $|\psi\rangle$ average to expectation value

$$\langle \hat{O} \rangle = \frac{\langle \psi | \hat{O} | \psi \rangle}{\langle \psi | \psi \rangle} = \langle \psi | \hat{O} | \psi \rangle$$

normalized $\langle \psi | \psi \rangle = 1$

$$\langle \hat{O} \rangle = \sum_{ij} \langle \psi | \theta_i \rangle \langle \theta_i | \hat{O} | \theta_j \rangle \langle \phi_j | \psi \rangle$$

$$= \sum_i \theta_i \left| \langle \theta_i | \psi \rangle \right|^2$$

classical probability

Commuting observables $\hat{A}, \hat{B}$ are compatible. They are simultaneously diagonalizable:

$$\hat{A} |a, b\rangle = a |a, b\rangle$$
$$\hat{B} |a, b\rangle = b \cdot |a, b\rangle$$

$$[A, B] |a, b\rangle = (\hat{A}\hat{B} - \hat{B}\hat{A}) |a, b\rangle = 0$$

probability of obtaining eigenvalue a, followed by b is same as b, followed by a

$$|\psi\rangle = \sum |ab\rangle\langle ab|\psi\rangle$$

$$P(a, b) = |\langle ab|\psi\rangle|^2 = P(b, a)$$

<u>Degeneracy</u> Suppose eigenvalues of $\hat{A}, \hat{B}$ can take on only 3 values but $b_1 = b_2$: states

$|a_1 b_1\rangle, \; |a_2 b_1\rangle, \; |a_3, b_3\rangle$

Consider general superposition state

$$|\psi\rangle = \lambda_1 |a_1, b_1\rangle + \lambda_2 |a_2, b_1\rangle + \lambda_3 |a_3, b_3\rangle$$

Since $[\hat{A}, \hat{B}] = 0$, must get same result for probability measured in either order.

<u>First</u>, measure $\hat{A}$ followed by $\hat{B}$, and suppose $\hat{A}$ gives eigenvalue a_1,

$$\hat{B}\hat{A}|\psi\rangle = \lambda_1 |a_1 b_1\rangle$$

$$P(b_1, a_1) = |\lambda_1|^2 \qquad |\psi\rangle \to |\psi'\rangle = |a_1, b_1\rangle$$

Now measure in reverse order

$$|\psi\rangle \xrightarrow{\hat{B}} |\psi'\rangle = \text{Const}\left[\lambda_1 |a_1 b_1\rangle + \lambda_2 |a_2 b_1\rangle\right]$$

$$= \frac{1}{\sqrt{|\lambda_1|^2 + |\lambda_2|^2}}\left[\lambda_1 |a_1 b_1\rangle + \lambda_2 |a_2 b_1\rangle\right]$$

with probability

$$\left|\left(\langle a_1, b_1| + \langle a_2, b_1|\right)|\psi\rangle\right|^2 = |\lambda_1|^2 + |\lambda_2|^2$$

then measure $\hat{A}$ and get a_1

$$|\psi'\rangle \xrightarrow{\hat{A}} |a_1, b_1\rangle$$

with probability

$$|\langle a_1, b_1|\psi'\rangle|^2 = \frac{|\lambda_1|^2}{\sqrt{|\lambda_1|^2 + |\lambda_2|^2}}$$

then $$P(a_1,b_1) = \frac{|\lambda_1|^2}{\sqrt{|\lambda_1|^2 + |\lambda_2|^2}} \cdot \left(|\lambda_1|^2 + |\lambda_2|^2\right)$$

$$= |\lambda_1|^2 = P(b_1, a_1)$$

A measurement of $\hat{B}$ only does not determine both eigenvalues, but measuring $\hat{A}$ & $\hat{B}$ does.

Generally, we can find a complete set of commutating operators whose eigenvalues give a unique state.

A set of observables $\{\hat{\Omega}_i\}$ is complete if

$$[\hat{\Omega}_i, \hat{\Omega}_j] = 0 \quad \text{commute}$$

and the state with corresponding eigenvalues $\hat{\Omega}_i|\psi\rangle = \omega_i|\psi\rangle$

that is $|\psi\rangle = |\omega_1, \omega_2, \ldots\rangle$ is non-degenerate.

This is called a complete set of commutating observables (CSCO). Once we measure this set, we know everything there is to know about the state of the system.

Example: hydrogen (without spin)

Complete set $\{\hat{H}, \hat{L}^2, \hat{L}_z\}$

eigenvalues

reduced mass $\approx m_e$

$$\hat{H}|n\ell m\rangle = -\frac{1}{n^2}\frac{\alpha^2 \mu c^2}{2}$$

$$\hat{L}^2 = \hbar^2 \ell(\ell+1)|n\ell m\rangle$$

$$\hat{L}_z|n\ell m\rangle = m|n\ell m\rangle$$

degeneracy of $\hat{H}$ is n^2

degeneracy of $\hat{L}^2$ is $2\ell+1$

Time dependence of expectation value.

$$\langle \hat{A} \rangle = \langle \psi(t) | \hat{A} | \psi(t) \rangle$$

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}} |\psi(0)\rangle$$

$$\frac{\partial}{\partial t}|\psi(t)\rangle = -i\frac{\hat{H}}{\hbar}|\psi(t)\rangle$$

giving

$$\frac{d\langle \hat{A} \rangle}{dt} = \langle \psi(t) | \left(\frac{i\hat{H}A}{\hbar} + \frac{\partial A}{\partial t} + \hat{A}\left(\frac{-iH}{\hbar}\right)\right) | \psi(t) \rangle$$

$$= \frac{i}{\hbar}\langle [H, A] \rangle + \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle$$

time dependence in state is called "Schrödinger picture."

We could put time dependence in operator "Heisenberg picture"

$$\langle \hat{A} \rangle = \langle \psi(0) | \underbrace{e^{\frac{i\hat{H}t}{\hbar}} \hat{A}\, e^{-iHt/\hbar}}_{= \hat{A}'} | \psi(0) \rangle$$

Giving operator equation:

$$\frac{dA'}{dt} = \frac{i}{\hbar}[H, A'] + \frac{\partial A'}{\partial t}$$