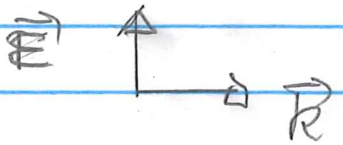
Lecture #5 Photon & Electron Angular Momentum

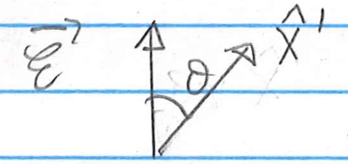
Photons are polarized just like classical EM waves



$\vec{\mathcal{E}} \cdot \vec{k} = 0$ transverse

We can create linearly polarized light with a polarizer. Intensity passing through second polarizer is then

$I_0 = |\vec{\mathcal{E}}|^2$



2nd polarizer at angle θ w.r.t. first.

$$I_1 = |\vec{\mathcal{E}} \cdot \hat{x}'|^2 = I_0 \cos^2\theta$$

polarization state of photons in a linear basis $|x\rangle, |y\rangle$

general polarization state

$$|\psi\rangle = \psi_x |x\rangle + \psi_y |y\rangle$$

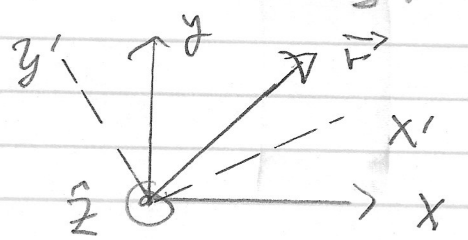
Polarization can also be written in circular basis

right / left $\quad |\pm\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle \pm i|y\rangle\right)$

Angular momentum operators $\vec{\hat{J}}$ are generators of rotations. Photon polarization is a Euclidean vector. Consider <u>Euclidean</u> rotation about the $\hat{z}$ axis

$$R^E(+\phi z) = \exp\left(-i\phi \frac{\hat{J}_z}{\hbar}\right)$$

right-handed rotation by ϕ about $\hat{z}$ axis.



passive rotation of $\vec{r}$

Polarization basis transforms as

$$(|x'\rangle, |y'\rangle) = (|x\rangle, |y\rangle)\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

components transform according to negative rotation

$$R^{\dagger}(+\phi\hat{z}) = R^{-1}(+\phi\hat{z}) = R(-\phi\hat{z})$$

$$\begin{pmatrix} r_x' \\ r_y' \end{pmatrix} = \begin{pmatrix} r_x \\ r_y \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

Consider circular polarization state

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle + i\,|y\rangle\right)$$

$$R^E(+\phi\hat{z})\,|+\rangle = \frac{1}{\sqrt{2}}\Big\{\cos\phi\,|x\rangle + \sin\phi\,|y\rangle + i\left(-\sin\phi\,|x\rangle + \cos\phi\,|y\rangle\right)\Big\}$$

$$= (\cos\phi - i\sin\phi)\frac{1}{\sqrt{2}}\left(|x\rangle + i|y\rangle\right)$$

$$= e^{-i\phi}|+\rangle$$

expand $\quad R^{E}(+\phi\hat{z}) = \exp\left(-i\phi\frac{\hat{J}_z}{\hbar}\right)$

$$= \left(I - i\phi\frac{J_z}{\hbar} + \cdots\right)$$

for infinitessimial angle ε, $e^{-i\varepsilon} \doteq 1 - i\varepsilon$

$$\left(I - i\varepsilon\frac{\hat{J}_z}{\hbar}\right)|+\rangle = (1 - i\varepsilon)|+\rangle$$

$$\frac{\hat{J}_z}{\hbar}|+\rangle = |+\rangle$$

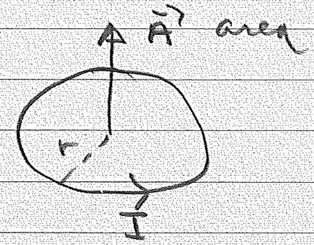
similarly $\quad \dfrac{\hat{J}_z}{\hbar}|-\rangle = -|-\rangle$

Eigenvalue of $\hat{J}_z$ for photon is ± 1.

Photon has spin one.

Stern Gerlach

classical charged particle with spin has magnetic moment



$$\vec{\mu} = \frac{I\vec{A}}{c}$$

$$I = \frac{qv}{2\pi r} \qquad A = \pi r^2$$

$$\mu_{loop} = \left(\frac{qv}{2\pi r}\right)\frac{\pi r^2}{c}\,\frac{m}{m} = \frac{q}{2}\frac{(mvr)}{mc} = \frac{1}{2}\frac{q}{mc}\left(S_{pin}\right)$$

more generally for classical spinning charged object there is a geometric factor

$$\vec{\mu}_{cl} = \frac{gq}{2mc}\vec{S}$$

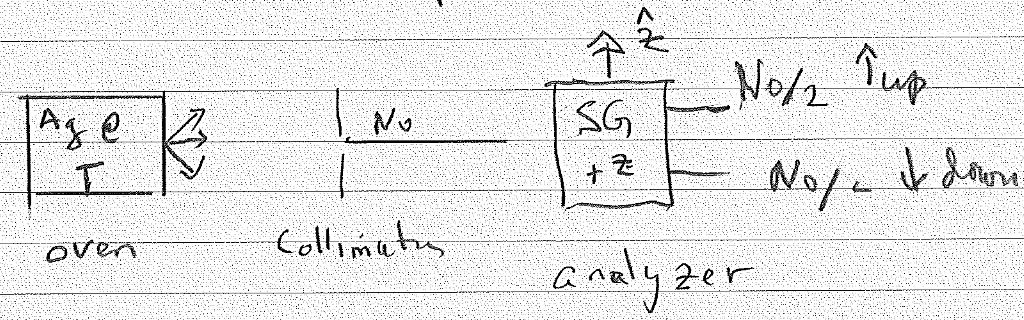
For electron or any structureless (point) particle, Dirac predicts $g = 2$.

Additional correction from quantum field theory

$$g \doteq 2(1+a)$$

$$a = \mathcal{O}(10^{-3}) \quad \text{measured, calculated}$$

Stern Gerlach Experiment



Silver $[Kr]\,4d^{10}\,5s^1$ atomic configuration

Ag spin = electron spin

effusion through small hole with Maxwell Boltzmann distribution

$$\overline{KE}_{hole} = 2k_BT \qquad \boxed{\vec{\mu}_e = \frac{-eg}{2m_ec}\vec{S}}$$

Electron in $\vec{B}$ field.

~~Homogeneous~~ $\quad \tau = \vec{\mu}\times\vec{B} = \dfrac{d\vec{S}}{dt} = \dfrac{-eg}{2mc}\vec{S}\times\vec{B} = \vec{\omega}\times\vec{S}$

Bohr magneton $\quad \mu^e_{Bohr} = \dfrac{\overset{2}{g}}{2}\dfrac{e}{m_ec}\hbar\underbrace{\left(\dfrac{\langle S_z\rangle}{\hbar}\right)}_{=1/2} = \dfrac{e\hbar}{2m_ec}$

$$= 5.79\times10^{-5}\ \mathrm{eV/T}$$

In B field with gradient

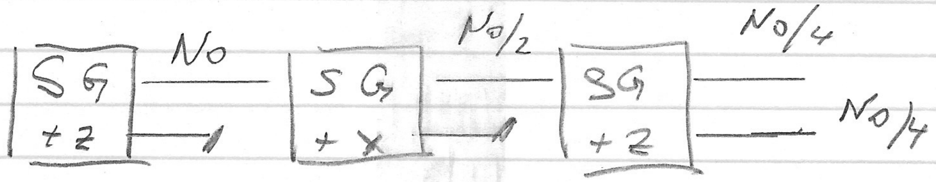
$$F_z = \vec{\mu}\cdot\left(\hat{z}\frac{\partial B}{\partial z}\right) = \frac{-eg}{2mc}\,\vec{S}\cdot\hat{z}\,\frac{\partial B}{\partial z}$$

Choose $\dfrac{\partial B}{\partial z} < 0$ so e spin ↑ deflects up for simplicity

Classically $\quad S_z = |\vec{S}|\cos\theta$

Stern Gerlach observe only $\quad S_z = \pm\frac{1}{2}\hbar$

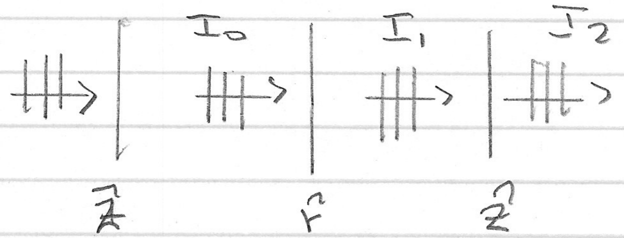
Electron spin eigenvalue $\pm\frac{\hbar}{2}$ with respect to any direction.



Outcome depends on order of SG measurements of $\hat{S}_x$, $\hat{S}_z$. Reverse these and get $\frac{N_0}{2}$!

Compare to photons (light)

intensities I



linear polarizers

$\vec{\mathcal{E}}_0 = \mathcal{E}_0 \hat{z}$ $\qquad$ $\vec{\mathcal{E}}_1 = \mathcal{E}_0 \cos\phi\, \hat{r}$ $\qquad$ $\vec{\mathcal{E}}_2 = \mathcal{E}_0 \cos^2\phi\, \hat{z}$

$I_0 = |\mathcal{E}_0|^2$ $\qquad$ $I_1 = \cos^2\phi\, I_0$ $\qquad$ $I_2 = \cos^4\phi\, I_0$

for $\hat{r}\cdot\hat{z} = 0$ $\left(\phi = \frac{\pi}{2}\right)$ $\quad I_1 = I_2 = 0$

Spin state of photon cannot be described by Euclidean polarization vector.

It is not a Euclidean vector, but a spinor.

A spinor is a two component, complex abstract vector

$$|\chi\rangle \doteq \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \qquad \langle\chi| \doteq (\chi_1^*, \chi_2^*)$$

The spinor rotates according to fundemental (defining) representation of group SU(2) S for det = +1, unitary, 2×2 matrices. Pauli matrices $\frac{\sigma_i}{2}$ are generators

$$R^S(\theta\hat{n}) = \exp\left(-i\theta\frac{\hat{n}\cdot\vec{\sigma}}{2}\right) \quad *$$

Almost all properties of group follow from

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\,\varepsilon_{ijk}\frac{\sigma_k}{2}$$

sum over repeated index k assumed, Einstein convention.

Electron spin operator $\hat{S}_i = \frac{\hbar}{2}\sigma_i$

Basis with eigenvalue $\pm\frac{\hbar}{2}$ measured by SG_z,

$$|+z\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix} \; ; \; |-z\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$$

* we found:

$$R^S(\theta\hat{n}) = \hat{I}\cos\frac{\theta}{2} - i\,\vec{\sigma}\cdot\hat{n}\sin\frac{\theta}{2}$$

Spinor Change of basis

$$|b_i\rangle = \hat{U}|a_i\rangle$$ $\hat{U}$ unitary operator, $i = 1, 2$

$$|b_i\rangle = \sum_{j=1}^{2} |a_j\rangle \underbrace{\langle a_j|\hat{U}|a_i\rangle}_{U^a_{ji}}$$

U^a_{ji} matrix in $|a_i\rangle$ basis

sum over j

Sum over row implies

$$\left(|b_1\rangle, |b_2\rangle\right) = \left(|a_1\rangle, |a_2\rangle\right)\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22}\end{pmatrix}^a$$

Components transform as $|x\rangle$ arbitrary spinor

$$|x\rangle = \sum_i |a_i\rangle\langle a_i|x\rangle$$

$$= \sum_{i,j} |b_j\rangle\langle b_j|a_i\rangle\langle a_i|x\rangle$$

with $\langle b_j| = \langle a_j|\hat{U}^+$ get

$$|x\rangle = \sum_j |b_j\rangle \left\{ \sum_i \underbrace{\langle a_j|\hat{U}^+|a_i\rangle}_{U^{+a}_{ji}}\langle a_i|x\rangle \right.$$

$$\langle b_k|x\rangle = \sum_j \underbrace{\langle b_k|b_j\rangle}_{\delta_{kj}} \sum_i U^{+a}_{ji}\langle a_i|x\rangle$$

$$= \sum_i U^{a+}_{ki}\langle a_i|x\rangle$$

sum over column

Components transform as ;

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^b = U^{a\dagger} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^a$$

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}^\dagger = \begin{pmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \end{pmatrix}$$

<u>Transformation of operators</u> :

$$|b_i\rangle = \hat{U}|a_i\rangle$$

$$\langle b_i|\hat{O}|b_j\rangle = \langle a_i|\hat{U}^\dagger \hat{O}\hat{U}|a_j\rangle$$

$$= \sum_{k,l} \langle a_i|\hat{U}^\dagger|a_k\rangle\langle a_k|\hat{O}|a_l\rangle \times \langle a_l|\hat{U}|a_j\rangle$$

in matrix form

$$[\hat{O}]^b_{ij} = [U^\dagger]^a_{ik}[\hat{O}]^a_{kl}[\hat{U}]_{lj}$$

or $$\hat{O}^b = \hat{U}^\dagger \hat{O}^a \hat{U}$$

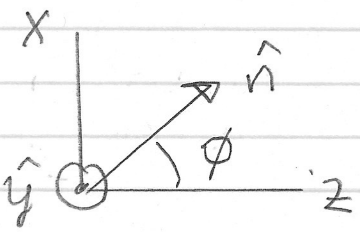
Electron spin precession in constant, uniform magnetic field $\vec{B}$.

$$\hat{H} = -\vec{\mu}\cdot\vec{B} = -\left(\frac{-eg\hbar}{2mc}\right)\frac{\vec{S}}{\hbar}\cdot\vec{B}$$

with $\vec{B} = B\hat{z}$, $\hat{H} = \hbar\omega\left(\frac{\sigma_z}{2}\right)$

$$\hbar\omega = \frac{egB}{2mc}\hbar = \mu_{Bohr} B$$

Suppose we prepare electron in spin state $|+n\rangle$



One way to solve this problem is to rotate the state $|+n\rangle$ into the $|\pm z\rangle$ basis.

For simplicity take $\phi = \frac{\pi}{2}$*. Rotation $-\hat{y}\phi$ takes n-basis to z-basis. Coordinates rotate by $+\vec{y}\phi$.

$$|\chi(0)\rangle \underset{z}{\doteq} \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & +\cos\theta/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}^{z}$$

$$\underset{\theta=\pi/2}{=} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{z}$$

* $|+n\rangle \underset{\phi=\pi/2}{=} |+x\rangle$

$$|\chi(t)\rangle = e^{-i\hat{H}t/\hbar}|\chi(0)\rangle \doteq_{z} \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t/2} \\ e^{+i\omega t/2}\end{pmatrix}$$

rotate back to n basis

$$|\chi(t)\rangle \doteq_{n} \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t/2} \\ e^{i\omega t/2}\end{pmatrix}$$

oops!I used theta here instead of phi

$$\underset{\theta=\pi/2}{=} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t/2} \\ e^{i\omega t/2}\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{-i\omega t/2} + e^{+i\omega t/2} \\ -e^{-i\omega t/2} + e^{+i\omega t/2}\end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega t/2 \\ i\sin\omega t/2 \end{pmatrix}$$

$$|\chi(\tfrac{2\pi}{\omega})\rangle \doteq_{n} -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -|\chi\rangle$$

Spinor rotated by 2π multiplies state by -1.

probability $|\langle +x|\chi(t)\rangle|^2 = \cos^2\left(\frac{\omega t}{2}\right)$

$$|\langle -x|\chi(t)\rangle|^2 = \sin^2\omega t/2$$

Then $\langle \hat{S}_x(t)\rangle = +\frac{\hbar}{2}\cos^2\frac{\omega t}{2} - \frac{\hbar}{2}\sin^2\omega t/2$

$$= \frac{\hbar}{2}\cos(\omega t)$$

Corresponding to classical spin precession