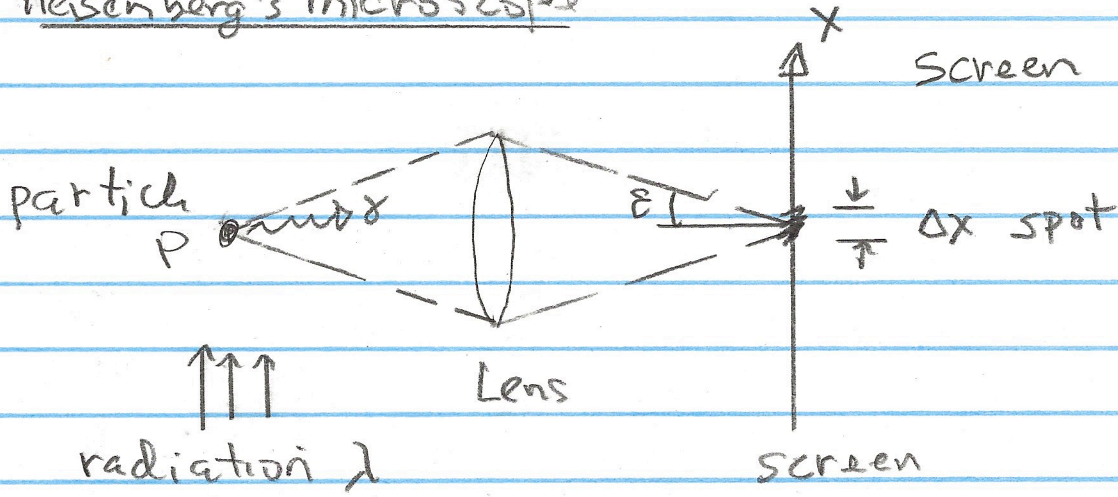
Lec 6: Uncertainty

Heisenberg's microscope



resolving power of lens $\Delta x \sim \frac{\lambda}{\sin \varepsilon}$

scattered photon $\Delta p_x \sim \frac{h}{\lambda} \sin \varepsilon$

uncertainty product $\Delta x \Delta p_x \sim h$

Operators X, $p_x = \frac{\hbar}{i} \frac{\partial}{\partial x}$

commutator operator identity

$$\left[x, \frac{\hbar}{i} \frac{\partial}{\partial x}\right] = i\hbar$$

rigorously gives $\Delta x \Delta p_x \geq \frac{\hbar}{2}$

General Uncertainty Relation:

Hamiltonian operators $\hat{A}, \hat{B}$

define $\hat{A}' = \hat{A} - \langle A \rangle$, $\hat{B}' \equiv \hat{B} - \langle B \rangle$

Schwartz inequality: $\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2$
applied to

$$|\alpha\rangle = \hat{A}'|\psi\rangle, \quad |\beta\rangle = \hat{B}'|\psi\rangle$$

where $|\psi\rangle$ is arbitrary state

$$\langle \psi | \hat{A}' \hat{A}' | \psi \rangle \langle \psi | \hat{B}' \hat{B}' | \psi \rangle \geq |\langle \psi | \hat{A}' \hat{B}' | \psi \rangle|^2$$

$$\langle A'^2 \rangle \langle B'^2 \rangle \geq |\langle \hat{A}' \hat{B}' \rangle|^2$$

now write

$$\hat{A}'\hat{B}' = \tfrac{1}{2}[\hat{A}', \hat{B}'] + \tfrac{1}{2}\{\hat{A}' \hat{B}'\}$$

where anticommutator $\{\hat{A}' \hat{B}'\} = \hat{A}'\hat{B}' + \hat{B}'\hat{A}'$

$\langle [A', B'] \rangle^* = -\langle [A', B'] \rangle$ pure imaginary

$\langle \{A', B'\} \rangle^* = +\langle \{A', B'\} \rangle$ pure real

and $[\hat{A}', \hat{B}'] = [\hat{A}, \hat{B}]$ so

$$\langle A'^2 \rangle \langle B'^2 \rangle \geq \tfrac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2 + \tfrac{1}{4} |\langle \{\hat{A}', \hat{B}'\} \rangle|^2$$

$\langle A'^2\rangle = (\Delta A)^2$; $\langle B'^2\rangle = (\Delta B)^2$

We have the inequality

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle [A,B]\rangle\right|$$

in the case of $\hat{X}, \hat{P}_x$ we have $\Delta X \Delta P_x \geq \hbar/2$

equality holds if

$$\hat{A}'|\psi\rangle = c\,\hat{B}'|\psi\rangle$$ c=ib, b real constant

$$\langle\psi|\{\hat{A},\hat{B}\}|\psi\rangle = 0$$

Note for Gaussian wave packet:

$$\psi(x) = \langle X|\psi\rangle = \left(\frac{1}{2\pi\sigma}\right)^{1/4} e^{ikx}\, e^{-\frac{X^2}{4\sigma^2}}$$

$$\int \psi^*\psi\, dx = \int dx\; G(X;\sigma,0) = 1$$

$$\hat{P}_x\psi(x) = \left(\frac{\hbar}{i}\right) ik\psi - \left(\frac{\hbar}{i}\right)\frac{2X}{4\sigma^2}\psi$$

$$= \hbar k\psi + i\hbar\frac{X}{2\sigma^2}\psi$$

$\hat{P}_x' = P_x - \langle P_x\rangle$ gives

$$\hat{P}_x'\psi(x) = \left(i\hbar\frac{X}{2\sigma^2}\right)\psi \propto X\psi$$

On HW, you will show $\langle\{x,P_x\}\rangle = 0$ for Gaussian

<u>Implications:</u> Size of atom. Hydrogen

Bohr radius $a_0 =$ 0.05 nm. Proton size $\sim 10^{-15}$ m $= 10^{-6}$ nm?

On homework, we found the radial momentum operator $\hat{p}_r = \frac{\hbar}{i}\left(\frac{1}{r}\frac{\partial}{\partial r} r\right)$

Easy to show $[r, \hat{p}_r] = i\hbar \Rightarrow \Delta r \, \Delta p_r \geq \frac{\hbar}{2}$

Hydrogen ground state as no angular momentum.

$$H = \frac{p_r^2}{2m} - \frac{\alpha \hbar c}{r}$$

$\alpha = 1/137.035\ldots$

$\hbar c = 197$ eV.nm $= 197$ MeV.fm

take $p_r = \frac{\hbar}{r}$

minimize $E(r) = \frac{\hbar^2}{2m}\left(\frac{1}{r^2}\right) - \frac{\alpha \hbar c}{r}$

$$\left.\frac{dE}{dr}\right|_{r_m} = 0 = -\frac{\hbar^2}{m}\left(\frac{1}{r_m^3}\right) + \frac{\alpha \hbar c}{r_m^2}$$

$$r_m = \frac{\hbar c}{mc^2 \alpha} \quad \text{Bohr radius}$$

$$E_{min} = \frac{\hbar^2}{2m}\left(\frac{mc^2\alpha}{\hbar c}\right)^2 - \alpha \hbar c \left(\frac{mc^2\alpha}{\hbar c}\right) = -\frac{1}{2}\alpha^2 mc^2$$

Exact having taken $p_r = \frac{\hbar}{r}$ which is just a guess.

Physics is correct.

Time-Energy Uncertainty

time is not observable, but there is an uncertainty relation. Mandelstam + Tamm (1945)

Observable $\hat{A}$.

$$\Delta H \Delta A \geq \frac{1}{2} \left| [\hat{H}, \hat{A}] \right|$$

and

$$\frac{\partial \langle A \rangle}{\partial t} = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle$$

$\hat{A}$ no explicit time dependence

$$\Delta H \Delta A \geq \frac{\hbar}{2} \frac{\partial \langle \hat{A} \rangle}{\partial t}$$

defining $\Delta T \equiv \dfrac{\Delta A}{\frac{\partial \langle \hat{A} \rangle}{\partial t}}$ gives $\Delta H \Delta T \geq \frac{\hbar}{2}$

ΔT shortest time required for significant change in A

From Mandelstam, Tamm

Example: decay of excited state

projection operator $\hat{P}_n|\psi\rangle = |\psi_n\rangle\langle\psi_n|\psi\rangle$

$\hat{P}_n^2 = \hat{P}_n$ so $(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \langle p\rangle - \langle p\rangle^2$

notation, let $\bar{p} = \langle p\rangle$

$$\Delta H \sqrt{\bar{p} - \bar{p}^2} \geq \frac{\hbar}{2}\left|\frac{\partial \bar{p}}{\partial t}\right|$$

integrate $\quad \Delta H t \geq \frac{\hbar}{2}\int_0^t \frac{|dp|}{\sqrt{\bar{p}-\bar{p}^2}}$

$\bar{p} + q = 1 \qquad q \equiv$ not decayed

take $\bar{p}(0) = 1$, $q(0) = 0$

$$\bar{p} - \bar{p}^2 = q - q^2$$

$$\int_0^t \frac{dq}{\sqrt{q-q^2}} = -2\sin^{-1}\sqrt{1-q}\,\Big|_0^t$$

$$= 2\left[\frac{\pi}{2} - \sin^{-1}\sqrt{p}\right]$$

$$\frac{\Delta H t}{\hbar} \geq \frac{\pi}{2} - \sin^{-1}\sqrt{p}$$

for $t \leq \frac{\pi\hbar}{2\Delta H} \quad \left(\frac{\Delta H t}{\hbar} \leq \frac{\pi}{2}\right)$

$$\sqrt{p} \geq \sin\left(\frac{\pi}{2} - \frac{\Delta H t}{\hbar}\right) = \cos\left(\frac{\Delta H t}{\hbar}\right)$$

$$P \geq \cos^2\left(\frac{\Delta H\, t}{\hbar}\right)$$

decay $\bar{P} = e^{-t/\tau}$

half-life $t_{1/2}$

$$\frac{1}{2} \geq \cos^2\left(\frac{\Delta H\, t_{1/2}}{\hbar}\right)$$

implies $\theta = \dfrac{\Delta H\, t_{1/2}}{\hbar} \geq \pi/4$

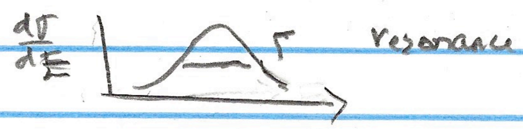
$$\Delta H\, t_{1/2} \geq \frac{\hbar \pi}{4}$$

$$\Delta H\, \tau \geq \hbar \frac{\pi}{4 \ln(2)}$$

we define natural line width ΔE by

$$\Delta E\, \tau \equiv \frac{\hbar}{2}$$

In measurement particle resonance
(too short to measure lifetime)



$\Delta E \equiv$ half width @ half maximum $= \Gamma/2$

for W boson $\Gamma = 2$ GeV

$$\tau_W = \frac{\hbar}{2}\frac{1}{\Gamma/2} = \frac{\hbar}{\Gamma} = \frac{\hbar c}{\Gamma}\left(\frac{1}{c}\right)$$

$\hbar c = 200$ MeV·fm giving

$$\tau_W = \frac{200 \text{ MeV}\cdot\text{fm}}{2000 \text{ MeV}} \frac{1}{3\times 10^{8+15}\ \text{fm/s}}$$

$$= \frac{1}{3} 10^{-25}\ \text{s}$$

Another example

in relativistic particle physics, particles can be created, annihilated in particle-antiparticle pairs.

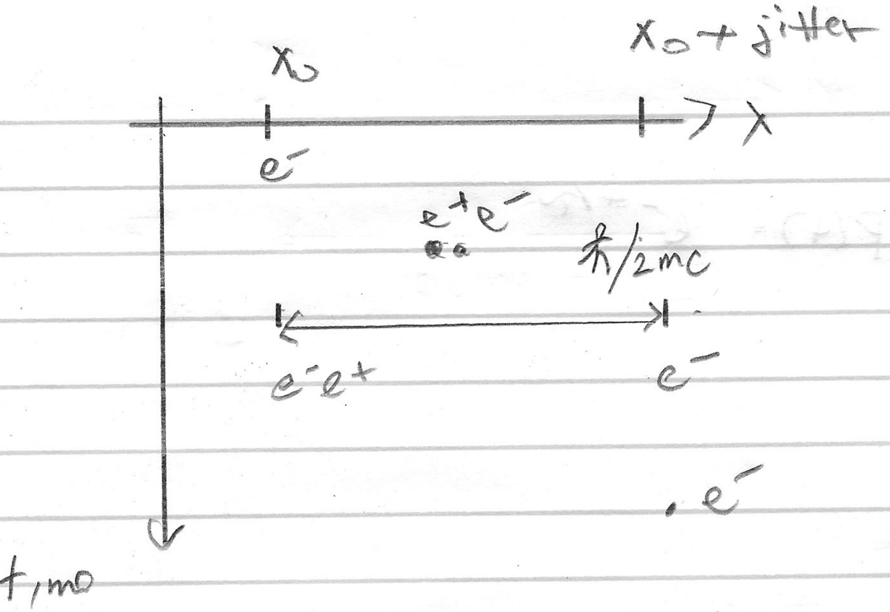
For electron, this gives rise to a jitter in position called zitterbewegung.

hand waving use of uncertainty — can create e^+e^- pair for a time

$$\Delta t = \frac{\hbar}{2}\left(\frac{1}{2mc^2}\right)$$

in this time e^+, e^- can travel a distance

$$c\Delta t = \frac{\hbar}{2}\left(\frac{1}{2mc}\right)$$



e^- at x_0
at $t=0$

$$\text{jitter} = \frac{\hbar}{2mc} = \frac{1}{2}\left(\frac{1}{\lambda_{\text{compton}}}\right)$$

numerically $\frac{\hbar c}{2mc^2} = \frac{200 \text{ MeV}\cdot\text{fm}}{2(0.5 \text{ MeV})} = 200 \text{ fm}$

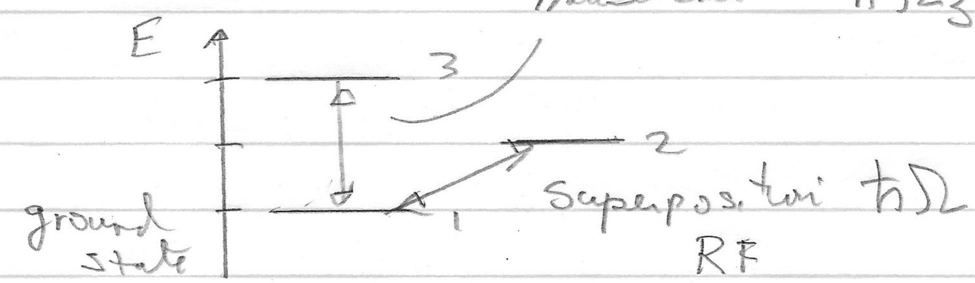
$$= 2\times10^{-4} \text{ nm}$$

this effect plays a role in relativistic corrections to the Hydrogen spectrum.

Quantum Zeno Itano et. al 1990

$^9Be^+$ ions in Penning trap (~ 5000)

transition from ground state to hyperfine levels split in magnetic field. Schematically



transition $3 \to 1$ allowed
transition $2 \to 1$ forbidden (metastable)

starting $|\psi(0)\rangle = |1\rangle$, applying RF

$$P_1(t) = \cos^2\left(\frac{\Omega t}{2}\right)$$
$$P_2(t) = \sin^2\left(\frac{\Omega t}{2}\right)$$

at $T = \frac{\pi}{\Omega}$, $P_1(T) = 0$, $P_2(T) = 1$

Applying measurement pulse of duration $\Delta \ll T$, observation of $3 \to 1$ transition means that ion is in state $|1\rangle$ at

If at $t = 0$ $|\psi(0)\rangle = |1\rangle$ measurement made at time $\tau \ll 1/\Omega$

$$P_1(\tau) \cong 1 \quad \& \quad P_2(\tau) \simeq \left(\frac{\Omega \tau}{2}\right)^2$$

Now, suppose starting at $t=0$ in state $|1\rangle$, we apply a sequence of measurement pulses at times

$$\tau_k = \frac{kT}{N} \qquad k = 1, \ldots N$$

Then Q.M. prediction is

$$P_2(T) = \frac{1}{2}\left[1 - \cos^N(\pi/N)\right]$$

$$\cong \frac{1}{2}\left[1 - \exp\left(\frac{-\pi^2}{2N}\right)\right] \xrightarrow[N \text{ large}]{} 0$$

For large N (~ 50), ion will never get to state $|2\rangle$!