Lec 9: Simple Harmonic Oscillator

Importance: Near potential minimum, almost all potentials are harmonic

$$V(X) = V(X_0) + \underbrace{(X-X_0)V'(X_0)}_{0 \text{ for } X_0 = \text{minimum}} + \frac{1}{2}(X-X_0)^2 V''(X_0) + \dots$$

as long as $V''(X_0) \neq 0$

$$V(x) - V(X_0) = \frac{1}{2}(X-X_0)^2 V''(X_0)$$

$$V''(X_0) = k = m\omega^2 \qquad \omega = \text{characteristic frequency} \times 2\pi$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

Define dimensionless variables

$$\hat{q} = \frac{\hat{p}}{\sqrt{m\hbar\omega}} \qquad \hat{y} = \hat{x}\sqrt{\frac{m\omega}{\hbar}}$$

turns out, classical turning point of ground state is $X_c = \sqrt{\frac{\hbar}{m\omega}}$ $\left(E_0 = \frac{\hbar\omega}{2}\right)$

$$\hat{y} = \frac{\hat{X}}{X_c} \; ; \qquad \hat{q} = \frac{X_c}{\hbar}\hat{p}$$

dimensionless energy $\varepsilon = \frac{E}{E_0} = \frac{2E}{\hbar\omega}$

note - different from Shankar

then $\hat{H} = \frac{1}{2}\hbar\omega(\hat{q}^2 + \hat{y}^2)$

$$[\hat{x}, \hat{p}] = i\hbar \Rightarrow [\hat{y}, \hat{q}] = i$$

implying in $|y\rangle$ basis $\quad \hat{q} = \frac{1}{i}\frac{\partial}{\partial y}$

time independent Schrodinger eq

$$\psi'' + (\varepsilon - y^2)\psi = 0 \qquad \left(\psi' \equiv \frac{\partial}{\partial y}\psi\right)$$

standard solution $|y| \to \infty$

$$\psi'' = y^2\psi \qquad \psi \xrightarrow[|y|\to\infty]{} e^{-y^2/2}$$

let $\psi = u(y)e^{-y^2/2}$

$$u'' - 2y u' + (\varepsilon - 1)u = 0$$

power series solution $\quad u(y) = \sum_{k=0}^{\infty} C_k y^k$

get recursion relation.

require $\quad \psi(y) \xrightarrow[|y|\to\infty]{} 0$

leads to termination of recursion relation and quantization of E.

$$E = \hbar\omega\left(\frac{1}{2} + n\right) \qquad n = 0, 1, 2, 3$$

the polynomials are standard Hermite polynomials

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) e^{-\frac{m\omega^2 x^2}{2\hbar}}$$

$H_0(y) = 1$

$H_1(y) = 2y$ (look them up)

$H_2(y) = 4y^2 - 2$

Ground state is Gaussian

generating function $H_n(y) = (-1)^n e^{y^2} \frac{\partial^n}{\partial y^n} e^{-y^2}$

series expansion $e^{-z^2+2zy} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(y)$

Algebraic Solution (Dirac) "Factorize"

$$\hat{a} \equiv \frac{1}{\sqrt{2}}(\hat{y} + i\hat{q})$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{y} - i\hat{q})$$

$$[\hat{y}, \hat{q}] = i \Rightarrow [\hat{a}, \hat{a}^\dagger] = 1$$

then $a^\dagger a = \frac{1}{2}(y - iq)(y + iq) = \frac{1}{2}\{y^2 + q^2 + i[\hat{y}, \hat{q}]\}$

$$= \frac{1}{2}(y^2 + q^2) - \frac{1}{2}$$

$$\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

$a^\dagger a$ is called the number operator $\hat{n}$.
define $\hat{n}$ basis

$$\hat{n}|n\rangle = n|n\rangle$$

$$\hat{H}|n\rangle = \hbar\omega(\hat{n}+\tfrac{1}{2})|n\rangle = \hbar\omega(n+\tfrac{1}{2})|n\rangle$$

So far, all we know is n is real

$$\hat{n}^\dagger = (a^\dagger a)^\dagger = a^\dagger a$$

Consider

$$[\hat{n},\hat{a}] = a^\dagger a a - \underbrace{a a^\dagger}_{a^\dagger a + 1} a$$

$$= a^\dagger a a - a^\dagger a a - a = -a$$

also

$$[\hat{n},\hat{a}]^\dagger = [a^\dagger,\hat{n}] = -a^\dagger$$

$$[\hat{n},\hat{a}^\dagger] = +a^\dagger$$

the state $a^\dagger|n\rangle$ .. Eigenvalue is

$$\hat{n}(a^\dagger|n\rangle) = (a^\dagger n + a^\dagger)|n\rangle = (n+1)(a^\dagger|n\rangle)$$

So $a^\dagger|n\rangle = C_+|n\rangle$

similarly $a|n\rangle = C_-|n-1\rangle$

$C_\pm$ normalization constants.

taking hermitian conjugate

$$\langle n | a = c_+^* \langle n |$$

then

$$\langle n | a a^+ | n \rangle = |c_+|^2 \langle n | n \rangle$$

properly normalize all eigenstates $\langle n | n \rangle = 1$

$$\langle n | a a^+ | n \rangle = |c_+|^2$$

$$\langle n | (a^+ a + 1) | n \rangle = |c_+|^2$$

$$n + 1 = |c_+|^2 \quad \text{can choose phase} = 1 \text{ so}$$

$$c_+ = \sqrt{n+1}$$

similarly, $c_- = \sqrt{n}$.

then $\quad |n+1\rangle = \dfrac{a^+}{\sqrt{n+1}} |n\rangle \quad$ properly normalized

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle \qquad |0\rangle \text{ ground state}$$

Must exist ground state, so that

$$a |0\rangle = 0$$

from $\quad a|n\rangle = \sqrt{n} \, |n-1\rangle \quad$ we see

eigenvalue of ground state is number 0.

That makes all eigenvalues integers.

$$n = 0, 1, 2, \ldots$$

In y representation:

$$\langle y|n\rangle = \frac{1}{\sqrt{2^n n!}}\left(y - \frac{d}{dy}\right)^n \langle y|0\rangle$$

from $a|0\rangle = 0$ we get

$$0 = \langle y|\hat{a}|0\rangle = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right)\langle y|0\rangle$$

$$\frac{d}{dy}\langle y|0\rangle = -y\langle y|0\rangle$$

simple, first order equation

$$\langle y|0\rangle = \frac{1}{\pi^{1/4}} e^{-y^2/2}$$

Generating formula for Hermite polynomials:

$$\left(y - \frac{d}{dy}\right)^n e^{-y^2/2} = H_n(y)\, e^{-y^2/2}$$

Cosmological Constant Problem

Each frequency mode of Em field is harmonic oscillator, with photons

$$a^\dagger_{\vec{k},\lambda}|0\rangle = |\vec{k},\lambda\rangle \qquad \lambda \text{ polarization}$$

Zero-point harmonic oscillator energy implies that each frequency mode has vacuum energy $\frac{1}{2}\hbar\omega = \frac{1}{2}\hbar c k$. Summing up gives

Vacuum energy density

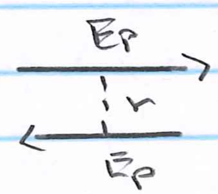
$$\langle \rho_{Em} \rangle = \int_0^\Lambda \frac{4\pi k^2 dk}{(2\pi)^3}\left(\frac{k}{2}\right) = \frac{\Lambda^4}{16\pi^2}$$

Where formally divergent integral is cut off at Planck scale when quantum field theory must break down,

$$\Lambda = \frac{1}{\sqrt{8\pi G}} \approx 10^{18}\ \text{GeV}$$

One way to think about this scale: proton Coulomb repulsion compared to gravitational attraction.
Gravity couples to energy ($E_p = M_p c^2 \gamma$)

2 protons with ultra-high energy



proton attracted as $\frac{E_p}{c^2}$. Equate Coulomb repulsion to gravitational attraction

$$\frac{\alpha \hbar c}{r} = \frac{G}{r}\left(\frac{E_p}{c^2}\right)^2$$

ignore the factor α, $\hbar c = G\left(\frac{E_p}{c^2}\right)^2$

$$E_p = c^2\sqrt{\frac{\hbar c}{G}} \quad \text{so} \quad \Lambda \propto \frac{1}{\sqrt{G}}$$

in suitable units $\hbar = 1$, $c = 1$ $\Lambda = \frac{1}{\sqrt{8\pi G}}$

then $\langle \rho_{EM} \rangle = 2 \times 10^{71}\ GeV^4$

observational value (CMB, ...)

$$\rho_{obs} = 10^{-47}\ GeV^4$$

No solution exists to solve this astounding discrepancy.