Phys 531

Fall 2023

Recitation #3 Solutions

(1) sequence of infinitessimal time evolutions,

$$\hat{U}(t,0) = \lim_{n\to\infty} U(n\Delta t,(n-1)\Delta t)\,\hat{U}((n-1)\Delta t,(n-2)\Delta t)\cdots U(\Delta t,0)$$

$$= \lim_{n\to\infty} \prod_{k=1}^{n} \hat{U}(k\Delta t,(k-1)\Delta t)$$

where $\Delta t = t/n$

$$\hat{U}(k\Delta t,(k-1)\Delta t) = I - \frac{i\Delta t}{\hbar}\hat{H}(k\Delta t)$$

$$= \exp\left(\frac{-i\Delta t}{\hbar}\hat{H}(k\Delta t)\right)$$

$$\hat{U}(t,0) = \lim_{n\to\infty} \prod_{k=1}^{n} \left(I - \frac{i\Delta t}{\hbar}\hat{H}(k\Delta t)\right)$$

$$= \lim_{n\to\infty} \prod_{k=1}^{n} \exp\left(\frac{-i\Delta t}{\hbar}\hat{H}(k\Delta t)\right)$$

$$= \lim_{n\to\infty} \exp\left\{\frac{-i}{\hbar}\sum_{k=1}^{\infty}\hat{H}(k\Delta t)\right\}$$

In limit $\Delta t = \frac{t}{n} \to dt'$ $\quad k\Delta t \to t'$

$$\hat{U}(t,0) = \exp\left[\frac{-i}{\hbar}\int_0^t dt'\,\hat{H}(t')\right]$$

(2) $$\int \underset{\substack{\uparrow \\ \text{constant}}}{\delta(x-x')}\, f(x')\, d\overset{\text{integration}}{x'} = f(x)$$

$$\delta'(x-x') \equiv \frac{d}{dx}\delta(x-x') = -\frac{d}{dx'}\delta(x-x')$$

to see this

$$\int \delta'(x-x') f(x') dx' = \frac{d}{dx}\int \delta(x-x') f(x') dx'$$

$$= \frac{d}{dx} f(x) = -\int \frac{d}{dx'}\delta(x-x') f(x') dx'$$

integrate by parts

$$= \int \delta(x-x') \frac{d f(x')}{dx'} dx' = \frac{d f(x)}{dx}$$

start with

$$\hat{p}\,|p\rangle = p\,|p\rangle$$ p-basis definition

$$\langle x|\hat{p}|p\rangle = p\langle x|p\rangle$$

insert $\int |x'\rangle\langle x'|\, dx'$

$$\int \langle x|\hat{p}\,|x'\rangle\langle x'|p\rangle = -i\hbar\int \delta'(x-x')\langle x'|p\rangle$$

$$= -i\hbar\frac{d}{dx}\langle x|p\rangle = p\langle x|p\rangle$$

$\phi_p(x) = \langle x|p\rangle$ plane wave

$$\frac{d}{dx}\langle x|p\rangle = \frac{ip}{\hbar}\phi_p(x)$$

integrate,

$$\langle x|p\rangle = C\, e^{ipx}$$

normalization

$$\langle p'|p\rangle = \delta(p'-p) = \int dx \langle p'|x\rangle\langle x|p\rangle$$

$$= |C|^2 \int dx \exp\left(i(p-p')x\right)$$

$$= |C|^2\, 2\pi\, \delta(p'-p) \quad \Rightarrow \quad C = \frac{1}{\sqrt{2\pi}}$$

so

$$\phi_p(x) = \frac{1}{\sqrt{2\pi}}\, e^{ipx}$$

<u>note that δ function is even</u>,

$$\delta(x) = \delta(-x)$$