phys 521, Fall 2023

Recitation # 5 Solutions

① For the spinor state

$$|\psi\rangle = A\left[3i\,|+z\rangle + 4\,|-z\rangle\right] \xrightarrow[+z]{} A\begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$\langle\psi|\psi\rangle = 1 = A^2(3^2+4^2) \Rightarrow A = \frac{1}{5}$$

$$\langle \hat{S}_y \rangle = \frac{1}{5^2}\,(-3i, 4)\,\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$= \frac{\hbar}{2}\frac{1}{5^2}\,(-3i, 4)\begin{pmatrix} -4i \\ -3 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{5^2}\,12(-1-1)$$

$$= -\hbar^2\,\frac{12}{25}$$

$$\hat{S}_y^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \langle S_y^2 \rangle = \frac{\hbar^2}{4}$$

$$(\Delta S_y)^2 = \langle S_y^2\rangle - \langle S_y\rangle^2 = \frac{\hbar^2}{4} - \hbar^2\underbrace{\left(\frac{12}{25}\right)^2}_{(0.48)^2 = 0.23} = +0.02\hbar^2$$

$$\Delta S_y = +0.14\hbar$$

② $$\hat{P}_+ = |+z\rangle\langle +z| \xrightarrow[+z]{} \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$\hat{P}_- = |-z\rangle\langle -z| \xrightarrow[+z]{} \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$\left(|+y\rangle, |-y\rangle\right) = \left(|+z\rangle, |-z\rangle\right) \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\i&-i\end{pmatrix}}_{\hat{S}}$$

$$[\hat{P}_+]^y = \hat{S}^\dagger [P_+]^z \hat{S}$$

$$= \frac{1}{2}\begin{pmatrix}1&-i\\1&i\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&1\\i&-i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&-i\\1&i\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

$$[\hat{P}_-]^y = \hat{S}^\dagger [\hat{P}_-]^z \hat{S} = \frac{1}{2}\begin{pmatrix}1&-i\\1&i\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1&1\\i&-i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&-i\\1&i\end{pmatrix}\begin{pmatrix}0&0\\i&-i\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$$

$$|+z\rangle \xrightarrow[+y]{} \hat{S}^\dagger\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$$

$$|-z\rangle \xrightarrow[+y]{} \hat{S}^\dagger\begin{pmatrix}0\\1\end{pmatrix} = \frac{-i}{2}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\hat{P}_+|+z\rangle \xrightarrow[+y]{} \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}\left(\frac{1}{2}\right)\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\hat{P}_-|+z\rangle \xrightarrow[+y]{} \frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\hat{P}_-|-z\rangle \xrightarrow[+y]{} \frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\left(\frac{-i}{2}\right)\begin{pmatrix}1\\-1\end{pmatrix} = \frac{-i}{2}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$P_+|-z\rangle \xrightarrow[+y]{} \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}\frac{-i}{2}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\hat{P}_+^2 \xrightarrow[+y]{} \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = [\hat{P}_+]^y$$

$$\hat{P}_-^2 \xrightarrow[+y]{} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = [\hat{P}_-]^y$$

$$\hat{P}_+\hat{P}_- \xrightarrow[+y]{} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{P}_-\hat{P}_+ \xrightarrow[+y]{} \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$