Quantum 521: TEST # 1

Please return the test with your work. No book, no notes, no google, no discussion. Calculator OK.

#1) A particle of mass m in a 1-D box, $0 < x < a$. Suppose that at time $t = 0$ the particle is in the state,

$$\psi(x) = \sqrt{30}a^p\Big[x(a-x)\Big]$$

where the normalization constant is $\sqrt{30}a^p$ where p is a real number.

a) What is p? Use a dimensional argument to avoid any integration.

b) What is the probability at $t = 0$ to measure the particle in the ground state of the box? Use, $\int_0^\pi [y(\pi - y)\sin y]\, dy = 4$.

c) Does this probability depend on time? Prove your answer.

#2) (a) Prove that for any normalized state $|\Psi\rangle$, $\langle\Psi|H|\Psi\rangle \geq E_0$ where E_0 is the lowest energy bound state eigenvalue.

(b) This result is the basis for the variational method: Given a wave function that depends on a parameter a (Ψ_a) minimizing the expectation value $\langle\Psi_a|H|\Psi_a\rangle = E_a$ with respect to the parameter a, the energy E_a must be greater than the true ground state energy. Use the variational method to prove that there exists at least one bound state for any attractive potential in one dimension. With $V(x \to \pm\infty) = 0$ the attractive potential is always negative, $V(x) = -|V(x)|$. Use the Gaussian wave function

$$\psi = \left(\frac{\mathrm{a}}{\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{\mathrm{ax}^2}{2}\right)$$

to show that the parameter a can always be chosen to make $E_a < 0$ and therefore the ground state energy must likewise be negative.

#3) (a) Consider a Hamiltonian that depends on some parameter λ with energy eigenstates

$$\hat{H}(\lambda)|\Psi_E\rangle = E(\lambda)|\Psi_E\rangle$$

where since E depends on λ the state $|\Psi_E\rangle$ does as well. Prove that

$$\frac{dE}{d\lambda} = \langle\Psi_E|\frac{d\hat{H}}{d\lambda}|\Psi_E\rangle$$

(b) Show the expectation value with respect to an energy eigenstate of any operator A, $\langle\Psi_E|A|\Psi_E\rangle$ where A has no explicit time dependence, is time independent.

(c) Now derive a relation starting from

$$\frac{d\langle\hat{x}\hat{p}\rangle}{dt}$$

Use this result together with the result of part (a) to find an explicit potential $V(x)$ with bound state energy splittings that scale with mass as $\Delta\mathrm{E} = \Delta\mathrm{m}/\mathrm{m} \times$ constant. Take the Hamiltonian to be $p^2/2m + V(x)$ (the mass appears only in the kinetic energy).