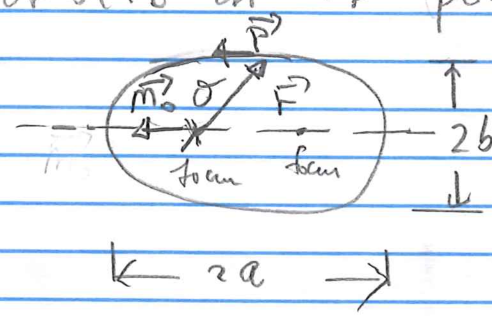
Phys 522 Spring 2025

Lec 2: Hydrogen Dynamical Symmetry

old but good reference on this is Schiff, Q.M.
also Commins
classical orbits in $1/r$ potential are closed.



a = semi-major

b = semi minor

eccentricity $\varepsilon = \sqrt{a^2 - b^2}/a$

classical Hamiltonian, energy is conserved

$$E = \frac{-e^2}{2a}$$

rotational symmetry $\vec{L} = \vec{r} \times \vec{p}$ = constant gives orbit in plane.

$$L^2 = \mu e^2 a (1 - \varepsilon^2)$$

closed orbit due to another constant vector

Runge-Lenz $\vec{M}_0 = \frac{\vec{p} \times \vec{L}}{\mu} - \frac{e^2}{r}\vec{r}$

$$|\vec{M}_0| = e^2 \varepsilon$$

$\vec{L} \cdot \vec{M}_0 = 0$ $\qquad M_0^2 = \frac{2H}{\mu^2} L^2 + e^4$

Corresponding Q.M. operator!

$$(\vec{p}\times\vec{L})_i^\dagger = (\varepsilon_{ijk} p_j L_k)^\dagger = \varepsilon_{ijk} L_k p_j$$
$$= -(\vec{L}\times\vec{p})_i \quad \text{not Hermitian}$$

$$\vec{m}_0 = \frac{1}{2\mu}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - \frac{e^2}{r}\vec{r}$$

$$[m_{0i}, H] = 0 \quad \text{conserved}$$

$$\vec{L}\cdot\vec{m}_0 = \vec{m}_0\cdot\vec{L} = 0 \qquad \vec{L}\perp\vec{m}_0$$

$$m_0^2 = \frac{2H}{\mu}(L^2+\hbar^2) + e^4$$

Pauli, 1926

"... considerable amount of computation" - Schiff
from commutation of $\vec{r}, \vec{p}$.

$\vec{m}_0$ are generators of "rotation" analogous to $\vec{L}$

$6!/2!4! = 15$ commutators

$$[L_i, L_j] = i\hbar\, \varepsilon_{ijk} L_k \qquad (3)$$

$$[m_{0i}, L_j] = i\hbar\, \varepsilon_{ijk} m_{0k} \qquad (3\times 3 = 9)$$

last three (hard!)

$$[m_{0i}, m_{0j}] = -\frac{2i\hbar}{\mu} H\, \varepsilon_{ijk} L_k \qquad (3)$$

Hamiltonian

Algebra of 6 generators does not close.
Work in subspace of Hilbert space with

$$H \psi_E = E \psi_E \qquad \underline{\underline{E < 0}} \quad \underline{\text{bound state}}$$

scale as $\quad \vec{m} = \left(\frac{-\mu}{2E}\right)^{1/2} \vec{m}_0$

Then last three commutators are

$$[m_i, m_j] = i\hbar \,\varepsilon_{ijk} L_k$$

So algebra in sub-space <u>closes</u>.

Identify with group $SO(4)$ orthogonal det $= 1$

4×4 real matrices. invent <u>fictitious</u> fourth coordinate r_4, P_4 with

$$[r_i, P_j] = i\hbar \,\delta_{ij} \qquad i, j = 1, 2, 3, 4$$

then define

$$m_1 \equiv X_1 P_4 - X_4 P_1$$

$$m_2 \equiv X_2 P_4 - X_4 P_2$$

$$m_3 \equiv X_3 P_4 - X_4 P_3$$

SO(4) is rank 2 group, 2 Casimir operators.

define

$$I_i = \frac{1}{2}(L_i + m_i)\ ;\ K_i = \frac{1}{2}(L_i - m_i)$$

get

$$[I_i, I_j] = i\hbar\,\varepsilon_{ijk} I_k$$

$$[K_i, K_j] = i\hbar\,\varepsilon_{ijk} K_k$$

$$[I_i, K_i] = 0 \quad [I_i, H] = 0 = [K_i, H]$$

then follows that 2 Casimir operators are

$$I^2 = \frac{1}{4}(\vec{L} + \vec{m})^2$$

$$K^2 = \frac{1}{4}(\vec{L} - \vec{m})^2$$

Since I_i, K_i commute they can be simultaneously diagonalized with eigenstates

$$\psi_{jk}$$

$$I^2 \psi_{jk} = j(j+1)\hbar^2 \psi_{jk}$$

$$K^2 \psi_{jk} = k(k+1)\hbar^2 \psi_{jk}$$

where $j, k = 0, \frac{1}{2}, 1, \ldots$ just as for SU(2)

Could have chosen Casimirs as

$$C = I^2 + k^2 \; ; \; C' = I^2 - k^2$$

$$C' = I^2 - k^2 = \frac{1}{2}\left(\vec{L}\cdot\vec{m} + \vec{m}\cdot\vec{L}\right)$$

since $\vec{L}\cdot\vec{m} = \vec{m}\cdot\vec{L} = 0$, $C' = 0$.

so $$C'\psi_{jk} = \hbar^2 \underbrace{\left[j(j+1) - k(k+1)\right]}_{=0} \psi_{jk}$$

must have $\underline{j = k}$.

Then for C, adding j, k terms

$$C\psi_k = 2k(k+1)\hbar^2 \psi_k \qquad k = 0, \tfrac{1}{2}, 1, \ldots$$

__However__

$$C = \frac{1}{2}\left(L^2 - \frac{\mu}{2E} m_0^2\right)$$

and $$m_0^2 = \frac{2E}{\mu}\left(L^2 + \hbar^2\right) + e^4$$

L^2 terms cancel, giving

$$C = -\frac{\mu e^4}{4E} - \frac{1}{2}\hbar^2$$

then from eigenvalue equation

$$2k(k+1)\hbar^2 = -\frac{\mu e^4}{4E} - \frac{1}{2}\hbar^2$$

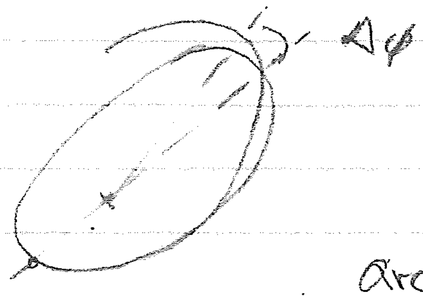
__now__ $$2k(k+1) + \frac{1}{2} = \frac{1}{2}(2k+1)^2$$

so $2k+1 = 1, 2, 3, \ldots n$ integer we define as n.

got energy,

$$E_n = \frac{-\mu e^4}{2\hbar^2 n^2}$$

Note on precession of Mercury



eccentricity of ellipse, greatly exaggerated

Perihelion	Arc-sec/century (wikipedia)
mercury observed	574
planet perturbations	532
G.R.	43
oblate sun	0.03

effect is tiny fraction of arc-second per century

but G.R., oblate both give correction to potential $V \sim \frac{1}{r^3}$

See Baierlein Newton Dynamics

As seen from Earth the precession of Mercury's orbit is measured to be 5600 seconds of arc per century (one second of arc=1/3600 degrees). Newton's equations, taking into account all the effects from the other planets (as well as a very slight deformation of the sun due to its rotation) and the fact that the Earth is not an inertial frame of reference, predicts a precession of 5557 seconds of arc per century. There is a discrepancy of 43 seconds of arc per century.